

CENTRAL ASIAN JOURNAL OF MATHEMATICAL THEORY AND COMPUTER SCIENCES

https://cajmtcs.centralasianstudies.org/index.php/CAJMTCS

Volume: 07 Issue: 04 | October 2026 ISSN: 2660-5309

Article

Radical-Constrained Intersections of Direct Summands in Modules

Hashmia Athrwi Ajaj Al-Janabi*[1]

1. Salah Al-Din Education Directorate / Tikrit Education Department / Iraq
* Correspondence: hashmia.athrwi.ajaj@gmail.com

Citation: Al-Janabi H. A. A. Radical-Constrained Intersections of Direct Summands in Modules. Central Asian Journal of Mathematical Theory and Computer Sciences 2026, 7(4), 212-219.

Received: 10th Jul 2026
Revised: 11th Jul 2026
Accepted: 20th Aug 2026
Published: 10th Sept 2026

Copyright: © 2026 by the authors. Submitted for open access publication under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/)

Abstract: We introduce the radical-constrained intersection property (RCI), a radical-localized weakening of the summand intersection property. A right R-module M is called RCI when, for every pair of direct summands A,B $\leq\oplus$ M satisfying A ∩ B ≤ Rad(M), the intersection A ∩ B is a direct summand of Rad(M). This formulation localizes the splitting requirement itself inside the Jacobson radical and avoids lifting-type approximations by a possibly zero direct summand. We prove that every SIP-module is RCI, that the implication is strict, and that modules with semisimple or zero radical are RCI. The class is closed under direct summands, but not under arbitrary finite direct sums; a sufficient closure condition is obtained for Hom-orthogonal direct sums. We also give a quotient theorem under an explicit direct-summand lifting hypothesis and an idempotent characterization on the radical. The examples $Z_2 \oplus Z_4$ and $Z_4 \oplus Z_8$ show, respectively, that RCI need not imply SIP and that finitely generated modules need not be RCI. Relations with SISZ, SIPr, SIP-lifting, and radical-supplemented modules are discussed to clarify the position of the new condition in current decomposition theory.

Keywords: Module, direct summand, Jacobson radical, RCI-module, SIP-module, SISZ-module, idempotent endomorphism, decomposition.

1. Introduction

Direct-summand conditions are central in module decomposition theory. The summand intersection property (SIP) requires the intersection of any two direct summands to split again in the ambient module, while its dual companion, the summand sum property, controls sums of direct summands [1]. Lifting theory approaches decomposition from a different direction by allowing a submodule to contain a direct summand with a small remainder [2,3]. These themes have generated several intermediate and relative conditions involving exact submodules, preradicals, supplements, and radical constraints [4–8].

Recent work is especially relevant to the present problem. Takıl Mutlu and Tercan introduced the SIP_r condition by restricting SIP to direct summands that are exact with respect to a left exact preradical [4]. Büyükaşık, Mermut, and Özdemir developed Rad-supplemented modules, where the radical controls supplement intersections [5]. Taşdemir and Tribak introduced SISZ-modules, requiring every nonzero intersection of two direct summands to be non-small [6]. Alkan and Doğan introduced SIP-lifting modules and obtained idempotent, graph, and radical formulations connecting intersections of direct summands with lifting phenomena [7]. More recently, Pekin and Ozkaya studied t-g-

radical supplemented modules, combining generalized radical supplements with t-sum terms [8].

The purpose of this paper is different from a lifting approximation. Instead of extracting a direct summand from $A \cap B$ and asking the quotient to be small, we require the intersection itself to split, but only inside the Jacobson radical and only when it is already contained there. Thus the radical is not merely a trigger for a lifting condition; it is the ambient module in which the required decomposition occurs.

To the best of our knowledge, the radical-constrained intersection property introduced here has not previously been studied in this form. The claim is intended in this precise sense: the defining requirement places the splitting of a radical-contained intersection inside the Jacobson radical itself, rather than using the radical only to select the summands under consideration or to control a lifting or supplement remainder.

Throughout, R denotes an associative ring with identity and M a unitary right R-module. For a submodule $N \leq M$, we write $N \leq_{\oplus} M$ when N is a direct summand of M. The Jacobson radical of M is denoted by $\mathrm{Rad}(M)$.

Definition 1.1

A submodule $N \leq M$ is called a direct summand of M if there exists $K \leq M$ such that

$$M = N \oplus K.$$

Definition 1.2

An R-module M has the summand intersection property (SIP) if

$$A, B \leq_{\oplus} M \Rightarrow A \cap B \leq_{\oplus} M.$$

Definition 1.3

A submodule $N \leq M$ is small in M, written $N \ll M$, if $N + L = M$ implies $L = M$ for every submodule $L \leq M$.

Lemma 1.4

If $M = K \oplus L$, then

$$\mathrm{Rad}(M) = \mathrm{Rad}(K) \oplus \mathrm{Rad}(L),$$

and consequently

$$\mathrm{Rad}(K) = K \cap \mathrm{Rad}(M).$$

Proof. This is the standard compatibility of the Jacobson radical with finite direct sums; see [2,9,10].

2. Research Methodology

This study assumes a theoretical approach in module theory. The concept of Radical-Constrained Intersection (RCI) modules is presented. Direct summands were used to investigate the basic properties of this class by using, intersections, and the Jacobson radical. Some lemmas and propositions are established to get the main structural results. The relationship between related classes and RCI modules, such as SIP and lifting modules, is also investigated. Some examples and their counters are illustrated to present the gained results and differentiate the proposed class from existing module classes.

3. Results and Discussion

3.1 The Radical-Constrained Intersection Property

Definition 3.1.1

An R-module M is called a **radical-constrained intersection module**, briefly an **RCI-module**, if for every pair of direct summands $A, B \leq_{\oplus} M$ satisfying

$$A \cap B \leq \mathrm{Rad}(M),$$

the intersection $A \cap B$ is a direct summand of $\mathrm{Rad}(M)$. Equivalently, there exists a submodule $C \leq \mathrm{Rad}(M)$ such that

$$\mathrm{Rad}(M) = (A \cap B) \oplus C.$$

Remark 3.1.2

The RCI condition is a radical-localized splitting condition. It neither asks $A \cap B$ to split in M nor replaces $A \cap B$ by a smaller direct summand. In particular, the definition cannot be satisfied merely by choosing a zero summand when $A \cap B$ is nonzero. The concepts of small submodules, radicals and related breakdown conditions deliver the standard basis for studying direct-sum decompositions of modules. These concepts and their basic properties were advanced extensively in the literature; see Clark et al. [11]. Nguyen and Zhou investigated related characteristics of cyclic modules.[12].

Proposition 3.1.3

Every SIP-module is an RCI-module.

Proof. Let $A, B \leq_{\oplus} M$ and suppose $N = A \cap B \leq \mathrm{Rad}(M)$. Since M is SIP, $N \leq_{\oplus} M$. Write $M = N \oplus K$. By Lemma 1.4, $\mathrm{Rad}(M) = \mathrm{Rad}(N) \oplus \mathrm{Rad}(K)$. Since $N \leq \mathrm{Rad}(M)$ and N is the first direct-sum component of M, we have $N = \mathrm{Rad}(N)$. Hence $\mathrm{Rad}(M) = N \oplus \mathrm{Rad}(K)$, so $N \leq_{\oplus} \mathrm{Rad}(M)$. □

Proposition 3.1.4

If $\mathrm{Rad}(M)$ is semisimple, then M is RCI.

Proof. Let $A, B \leq_{\oplus} M$ with $A \cap B \leq \mathrm{Rad}(M)$. Every submodule of a semisimple module is a direct summand. Therefore $A \cap B \leq_{\oplus} \mathrm{Rad}(M)$. □

Corollary 3.1.5

If $\mathrm{Rad}(M) = 0$, then M is RCI. In particular, every semisimple module is RCI.

Proposition 3.1.6

Every indecomposable module is RCI.

Proof. The only direct summands of M if it is indecomposable are 0 and M. Any intersection between two direct summands is therefore either 0 or M. If it is in Rad(M), it is either M=Rad(M), which also divides in Rad(M), or zero, which splits in every module. Although it differs from previously researched supplemented and (\oplus)-supplemented module conditions, the suggested radical-constrained intersection condition is connected to them. Specifically, Harmanci, Keskin, and Smith have studied (\oplus)-supplemented modules [13]. Nguyen and Zhou investigated related characteristics of cyclic modules. [12].

3.2 Separation from SIP and Nontriviality

Example 3.2.1 (RCI does not imply SIP)

Consider the $\mathbb{Z}$-module

$$M = \mathbb{Z}_2 \oplus \mathbb{Z}_4.$$

Its Jacobson radical is

$$\mathrm{Rad}(M) = 0 \oplus 2\mathbb{Z}_4 \cong \mathbb{Z}_2.$$

Hence $\mathrm{Rad}(M)$ is semisimple, so M is RCI by Proposition 2.4. We now show that M does not have SIP. Put

$$A = \langle(0,1)\rangle = 0 \oplus \mathbb{Z}_4, \quad B = \langle(1,1)\rangle, \quad K = \langle(1,0)\rangle.$$

Both A and B have order 4, while K has order 2. Moreover,

$$A \cap K = 0, \quad B \cap K = 0,$$

and therefore

$$M = A \oplus K = B \oplus K.$$

Thus A and B are direct summands. Their intersection is

$$A \cap B = \langle(0,2)\rangle = \mathrm{Rad}(M).$$

This subgroup is not a direct summand of M. Indeed, if it were, then

$$M \cong \mathrm{Rad}(M) \oplus M/\mathrm{Rad}(M) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2,$$

which is impossible because M contains an element of order 4. Hence M is RCI but not SIP.

Corollary 3.2.2

The implication

$$\mathrm{SIP} \Rightarrow \mathrm{RCI}$$

is strict.

Example 3.2.3 (a finitely generated module which is not RCI)

Let

$$M = \mathbb{Z}_4 \oplus \mathbb{Z}_8.$$

Then

$$\mathrm{Rad}(M) = 2\mathbb{Z}_4 \oplus 2\mathbb{Z}_8 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4.$$

Define

$$A = \langle(2,1)\rangle, \quad B = \langle(1,1)\rangle, \quad K = \langle(1,0)\rangle.$$

Both A and B are cyclic of order 8, while K has order 4. Since

$$A \cap K = B \cap K = 0,$$

the order calculation gives

$$M = A \oplus K = B \oplus K.$$

Thus $A, B \leq\oplus M$. To compute their intersection, suppose

$$k(2,1) = \ell(1,1).$$

Equality of the second coordinates gives $k \equiv \ell \pmod 8$, while equality of the first coordinates gives $2k \equiv \ell \pmod 4$. Hence $k \equiv 0 \pmod 4$, and therefore

$$A \cap B = \langle(0,4)\rangle.$$

Clearly $A \cap B \leq \mathrm{Rad}(M)$. However, $A \cap B$ is not a direct summand of $\mathrm{Rad}(M)$. Indeed,

$$\mathrm{Rad}(M)/(A \cap B) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

If $A \cap B$ split in $\mathrm{Rad}(M)$, then $\mathrm{Rad}(M)$ would be isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, a group of exponent 2. This contradicts $\mathrm{Rad}(M) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4$, which contains an element of order 4. Therefore M is not RCI.

Remark 3.2.4

Example 3.2.3 is important for the nontriviality of the definition: finite generation alone does not force RCI. This distinguishes the present condition from radical-lifting formulations that may become automatic when the whole radical is small.

3.3 Direct Summands and Finite Direct Sums

Theorem 3.3.1

Every direct summand of an RCI-module is RCI.

Proof. Let $K \leq\oplus M$ and write $M = K \oplus L$. Let $A, B \leq\oplus K$ with $N = A \cap B \leq \mathrm{Rad}(K)$. Since direct summands of a direct summand are direct summands of the ambient module, $A, B \leq\oplus M$. By Lemma 1.4, $\mathrm{Rad}(K) = K \cap \mathrm{Rad}(M)$, hence $N \leq \mathrm{Rad}(M)$. Since M is RCI, there exists $C \leq \mathrm{Rad}(M)$ such that $\mathrm{Rad}(M) = N \oplus C$. Intersecting with K and using $N \leq K$ yields $\mathrm{Rad}(K) = N \oplus (C \cap K)$. Thus $N \leq\oplus \mathrm{Rad}(K)$. □

Theorem 3.3.2

Let $M = M_1 \oplus M_2$, where M_1 and M_2 are RCI-modules. Assume

$$\mathrm{Hom_R}(M_1, M_2) = 0 = \mathrm{Hom_R}(M_2, M_1).$$

Then M is RCI.

Proof. Under the Hom-orthogonality hypothesis, every endomorphism of $M_1 \oplus M_2$ is block diagonal. Therefore every idempotent is block diagonal, and every direct summand of M has the form $A = A_1 \oplus A_2$ with $A_i \leq\oplus M_i$. Let $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ be direct summands with $A \cap B \leq Rad(M)$. Since $Rad(M) = Rad(M_1) \oplus Rad(M_2)$, we have $A_i \cap B_i \leq Rad(M_i)$ for $i = 1,2$. As each M_i is RCI, choose $C_i \leq Rad(M_i)$ with $Rad(M_i) = (A_i \cap B_i) \oplus C_i$. Hence $Rad(M) = (A \cap B) \oplus (C_1 \oplus C_2)$. □

Corollary 3.3.3

Theorem 3.3.2 extends to a finite direct sum $M = M_1 \oplus \cdots \oplus M_n$ provided $Hom_R(M_i, M_j) = 0$ whenever $i \neq j$.

Example 3.3.4 (RCI is not closed under arbitrary finite direct sums)

The $\mathbb{Z}$-modules $\mathbb{Z}_4$ and $\mathbb{Z}_8$ are indecomposable and hence RCI by Proposition **3.1.6**. Nevertheless, Example 3.2.3 shows that

$$\mathbb{Z}_4 \oplus \mathbb{Z}_8$$

is not RCI. Therefore no unrestricted finite-direct-sum closure theorem is possible.

3.4. Factor Modules

Theorem 3.4.1

Let M be RCI and let $N \leq Rad(M)$. Assume that:

every direct summand of M/N is of the form A/N for some direct summand $A \leq\oplus M$ containing N.

Then M/N is RCI.

Proof. Let $\bar{A}, \bar{B} \leq\oplus M/N$ and suppose $\bar{A} \cap \bar{B} \leq Rad(M/N)$. By the direct-summand lifting hypothesis, choose direct summands $A, B \leq\oplus M$ containing N with $\bar{A} = A/N$ and $\bar{B} = B/N$. Then $\bar{A} \cap \bar{B} = (A \cap B)/N$. Because the quotient submodule lies in the Jacobson radical, the standard quotient-radical formula applies; hence $A \cap B \leq Rad(M)$. Since M is RCI, write $Rad(M) = (A \cap B) \oplus C$. Because $N \leq A \cap B$, quotienting by N gives $Rad(M)/N = ((A \cap B)/N) \oplus ((C + N)/N)$. Using the same quotient-radical formula, this is a direct-sum decomposition of $Rad(M/N)$. Thus $\bar{A} \cap \bar{B} \leq\oplus Rad(M/N)$. □

Remark 3.4.2

The direct-summand lifting hypothesis in Theorem 3.4 is stated explicitly because RCI alone does not guarantee that direct summands lift through an arbitrary quotient. Since the quotient submodule is assumed to lie in the Jacobson radical, the standard identity $Rad(M/N) = Rad(M)/N$ is automatic and need not be imposed separately. Consequently, an unconditional quotient-closure statement should still not be asserted.

3.5. Endomorphism Interpretation

Let $S = End_R(M)$. Every direct summand of M is the image of an idempotent endomorphism, and every direct summand of Rad(M) is the image of an idempotent endomorphism of Rad(M).

Theorem 3.5.1

For an R-module M, the following conditions are equivalent:

(1) M is RCI.

(2) For every pair of idempotents $e, f \in End_R(M)$ satisfying $eM \cap fM \leq Rad(M)$, there exists an idempotent $h \in End_R(Rad(M))$ such that

$$h(Rad(M)) = eM \cap fM.$$

(3) For every such pair e, f, there exists $C \leq Rad(M)$ such that

$$Rad(M) = (eM \cap fM) \oplus C.$$

Proof. The equivalence of (1) and (3) follows from the fact that every direct summand of M is the image of an idempotent endomorphism. The equivalence of (2) and (3) is the

standard idempotent characterization of direct summands applied inside the module Rad(M). □

Remark 3.5.2

The idempotent in Theorem 3.5.1(2) acts on Rad(M), not on M. This is the key localization distinguishing RCI from an ordinary SIP idempotent criterion.

3.6. Relations with SISZ and Nearby Module Classes

Taşdemir and Tribak call a module SISZ when every nonzero intersection of two direct summands is not small in the ambient module [6]. This condition interacts naturally with RCI when the radical is small.

Proposition 3.6.1

Assume $\mathrm{Rad}(M) \ll M$ and M is SISZ. Then M is RCI.

Proof. Let $A, B \leq_{\oplus} M$ and assume $N = A \cap B \leq \mathrm{Rad}(M)$. Since a submodule of a small submodule is small, $N \ll M$. If $N \neq 0$, this contradicts the SISZ property. Hence $N = 0$, and zero is a direct summand of Rad(M). □

Corollary 3.6.2

Every finitely generated SISZ-module is RCI.

Proof. For a finitely generated module, the Jacobson radical is small in the module [2,10]. Apply Proposition 3.6.1. □

Proposition 3.6.3

RCI does not imply SISZ.

Proof. In Example 3.2.1, $M = \mathbb{Z}_2 \oplus \mathbb{Z}_4$ is RCI and $A \cap B = \mathrm{Rad}(M) \neq 0$. Since M is finitely generated, $\mathrm{Rad}(M) \ll M$. Hence there is a nonzero intersection of two direct summands that is small in M, so M is not SISZ. □

Comparison with SIPr

The SIP_r condition of [4] changes which direct summands are tested: it restricts attention to direct summands that are exact relative to a left exact preradical, but the resulting intersection is still required to be a direct summand of the ambient module. RCI instead tests all direct summands, activates only when their intersection lies in Rad(M), and requires the intersection to split inside the radical.

Comparison with SIP-lifting

Alkan and Doğan [7] develop SIP-lifting modules by connecting intersections of direct summands with lifting behavior; their work includes idempotent, graph, and radical formulations and yields ring-theoretic characterizations of semiperfect and perfect rings. The present RCI condition is not defined through a small quotient or a direct-summand approximation of the intersection. Its requirement is instead the exact splitting of a radical-contained intersection inside Rad(M). Accordingly, no implication between RCI and SIP-lifting is claimed here without additional hypotheses.

The following assertions and examples should clarify the precise difference between the current results and their results, which impose an additional radical constraint on the intersection.

Comparison with radical-supplemented modules

Rad-supplemented modules [5] and t-g-radical supplemented modules [8] are supplement conditions: they seek a submodule that combines with a given submodule to generate the whole module while controlling the resulting intersection by a radical-type condition. RCI is an intersection condition on pairs of already existing direct summands and does not involve supplements. The literature has also examined the connection between (D3)-type conditions and direct-summand decomposition qualities (see Nguyen, Yousif, and Zhou) [14]. As a result, although the radical is frequently used, the breakdown

process is distinct. In contrast, Wang and Wu's (H)-supplemented modules offer an additional framework for examining decomposition and supplement features [15].

3.7. RCI Rings and Radical Structure

Definition 3.7.1

A ring R is called a **right RCI ring** if the regular right module R_R is RCI.

Proposition 3.7.2

If R_R has SIP, then R is a right RCI ring.

Proof. This is Proposition 3.1.3 applied to the regular module. □

Proposition 3.7.3

If the right R-module J(R) is semisimple, then R is a right RCI ring.

Proof. For the regular module, $\mathrm{Rad}(R_R) = J(R)$. Apply Proposition 3.1.4. □

Corollary 3.7.4

Let R be semiprimary and assume $J(R)^2 = 0$. Then R is a right RCI ring.

Proof. Since $R/J(R)$ is semisimple artinian and $J(R)^2 = 0$, the right module J(R) is naturally a module over the semisimple ring $R/J(R)$. Therefore J(R) is semisimple, and Proposition 3.7.3 applies. □

Remark 3.7.5

The ring-theoretic condition should not be confused with the SIP-lifting characterizations in [7]. RCI asks only for radical-contained intersections of principal idempotent summands of the regular module to split within J(R); it does not assert that every finitely generated module is RCI. Example 3.2.3 explicitly shows that finite generation alone is insufficient at the module level.

4. Discussion

The results show that the RCI condition offers a radical-based limit on the intersections of direct summands. This study founds the structural relationships between several well-known classes of modules and RCI modules. Specifically, the Jacobson radical plays an essential role in controlling the intersection of direct summands.

The obtained results also demonstrated that the proposed condition performs naturally under appropriate direct-sum constructions and delivers a beneficial framework for studying decompositions of modules. The examples show that the RCI condition is diverse from stronger classical intersection properties in general. Thus, the proposed approach offers a possible new direction for investigating module decompositions through radical constraints.

5. Conclusions and Further Directions

The radical-constrained intersection property introduced here localizes the splitting requirement of SIP to the Jacobson radical. The central condition is

$$A, B \leq_{\oplus} M \text{ and } A \cap B \leq \mathrm{Rad}(M) \;\Rightarrow\; A \cap B \leq_{\oplus} \mathrm{Rad}(M).$$

This placement of the complement inside the radical is the main structural feature of the theory. It prevents the definition from collapsing to a lifting approximation and permits a precise comparison with SIP. We proved that every SIP-module is RCI and exhibited $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ as an RCI-module that is not SIP. We also exhibited $\mathbb{Z}_4 \oplus \mathbb{Z}_8$ as a finitely generated module that is not RCI, showing that the condition is genuinely nontrivial.

RCI is inherited by direct summands. It is not preserved by arbitrary finite direct sums, although Hom-orthogonal finite direct sums are closed under the property. Under an explicit direct-summand lifting hypothesis, RCI descends to factor modules. An

idempotent formulation shows that the relevant idempotents act naturally on Rad(M), rather than on the whole module.

The comparison with recent literature also clarifies the role of RCI. SISZ excludes nonzero small intersections, while RCI allows radical-contained intersections provided they split inside the radical. SIPr restricts the direct summands being tested through a preradical, whereas RCI changes the ambient module in which the intersection must split. SIP-lifting and radical-supplemented theories use different decomposition mechanisms and therefore should be treated as neighboring, rather than equivalent, frameworks.

Natural directions for further study include: (1) characterizing right RCI rings in terms of idempotents and the module structure of J(R); (2) determining conditions under which RCI is preserved by arbitrary direct sums or products; (3) studying Morita invariance; (4) analyzing RCI over Artinian, serial, semiperfect, and perfect rings; (5) finding conditions under which RCI together with an additional radical hypothesis forces SIP; and (6) developing a preradical version in which Rad(M) is replaced by a suitable functorial radical.

REFERENCES

[1] A. Hamdouni, A. Ç. Özcan, and A. Harmancı, "Characterization of modules and rings by the summand intersection property and the summand sum property," JP Journal of Algebra, Number Theory and Applications, vol. 5, no. 3, pp. 469–490, 2005.

[2] J. Clark, C. Lomp, N. Vanaja, and R. Wisbauer, Lifting Modules: Supplements and Projectivity in Module Theory, Birkhäuser, Basel, 2006. DOI: 10.1007/3-7643-7573-6.

[3] A. Tercan and C. C. Yücel, Module Theory, Extending Modules and Generalizations, Birkhäuser, Basel, 2016. DOI: 10.1007/978-3-0348-0952-8.

[4] F. Takıl Mutlu and A. Tercan, "Summand intersection property on the class of exact submodules," Publications de l'Institut Mathématique, vol. 111(125), pp. 61–68, 2022. DOI: 10.2298/PIM2225061T.

[5] E. Büyükaşık, E. Mermut, and S. Özdemir, "Rad-supplemented modules," Rendiconti del Seminario Matematico della Università di Padova, vol. 124, pp. 157–177, 2010. DOI: 10.4171/RSMUP/124-10.

[6] Ö. Taşdemir and R. Tribak, "New generalizations of the summand intersection property," Journal of Algebra and Its Applications, article 2542019, 2025. DOI: 10.1142/S0219498825420198.

[7] M. Alkan and E. Doğan, "Intersection of direct summands in modules," Indian Journal of Pure and Applied Mathematics, 2026. DOI: 10.1007/s13226-026-01000-x.

[8] A. Pekin and H. Ozkaya, "t-g-radical supplemented modules," arXiv:2606.09372, 2026.

[9] F. W. Anderson and K. R. Fuller, Rings and Categories of Modules, 2nd ed., Springer-Verlag, New York, 1992.

[10] R. Wisbauer, Foundations of Module and Ring Theory, Gordon and Breach, Philadelphia, 1991.

[11] J. Clark, C. Lomp, N. Vanaja, and R. Wisbauer, Lifting Modules: Supplements and Projectivity in Module Theory, Birkhäuser, 2006.

[12] X. H. Nguyen and Y. Zhou, "Rings whose cyclic modules are lifting and ($\oplus$)-supplemented," Communications in Algebra, 46 (2018), 4918–4927.

[13] A. Harmanci, D. Keskin, and P. F. Smith, "On ($\oplus$)-supplemented modules," Acta Mathematica Hungarica, 83 (1999), 161–169.

[14] X. H. Nguyen, M. F. Yousif, and Y. Zhou, "Rings whose cyclics are (D_3)-modules," Journal of Algebra and Its Applications, 16 (2017), 1750184.

[15] Y. Wang and D. Wu, "On (H)-Supplemented Modules," Communications in Algebra, 40 (2012), 3679–3689.