

Article

A Kind of Non-Commuting Graph Associated with the Derived Subgroup and Disjoint Two-Element Subsets of a Group

Yaqoob A. Farawi^{*1}

1. Department of Mathematics, University of Thi-Qar, Iraq

*Correspondence: yaqoob1980@utq.edu.iq

Abstract: This paper introduces a new graph associated with a subgroup of a finite group, whose vertices are the two-element subsets of the subgroup satisfying the prescribed condition, while adjacency is defined by a new relation between these subsets. The proposed graph provides a novel combinatorial model for studying subgroup structures from the perspective of algebraic graph theory. A connection between the proposed graph and the g -noncommuting graph introduced by Olue, Erfanian, and Jafarzadeh is established by proving that the connectivity of the proposed graph implies the connectivity of the corresponding g -noncommuting graph, whereas the converse does not hold in general. A counterexample is presented to demonstrate this fact. Finally, the proposed graph is applied to the dihedral group, and its main graph-theoretic properties are determined.

Keywords: Non-Commuting Graph, Derived Subgroup, Dihedral Group, Diameter, Girth, Connected Graph

Citation: Farawi, A. W. A Kind of Non-Commuting Graph Associated with the Derived Subgroup and Disjoint Two-Element Subsets of a Group. Central Asian Journal of Mathematical Theory and Computer Sciences 2026, 7(4), 413-417.

Received: 05th Aug 2026

Revised: 30th Aug 2026

Accepted: 25th Sep 2026

Published: 20th Oct 2026



Copyright: © 2026 by the authors. Submitted for open access publication under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>)

1. Introduction

Algebraic graphs constitute an important area of research that combines graph theory with algebraic structures such as groups, rings, and semigroups. These graphs provide useful tools for studying algebraic properties through combinatorial techniques and have attracted considerable attention from many researchers. In recent years, the authors have contributed to this area through several studies concerning graphs associated with algebraic structures and their topological and graph-theoretical properties [1]-[14]. Among the most extensively studied algebraic graphs is the non-commuting graph of a group, introduced by Abdollahi, Akbari and Maimani in 2006 [1].

This graph has become an important tool for investigating the structure of non-abelian groups and has inspired numerous generalizations and related constructions. In 2014, Tolue, Erfanian and Jafarzadeh introduced the (g) -noncommuting graph associated with a fixed element (g) of a finite group [6]. This construction generalized the classical non-commuting graph by incorporating a distinguished group element into the adjacency relation [15]. Later, several properties of these graphs were investigated by various authors. In particular, Nasiri, Erfanian, Farawi and Muchtadi-Alamsyah studied graph associated with a fixed element and a subgroup of a group and established several properties of this graph [10].

2. Materials and Method

Motivated by the above developments, we introduce and study a new type of non-commuting graph associated with a fixed element and the derived subgroup of a group. Unlike the previously known constructions, the vertices of our graph consist of pairwise disjoint binary subsets rather than individual group elements. The adjacency relation is defined through the fixed element and the derived subgroup, producing a graph that captures new algebraic information about the underlying group. We investigate several fundamental properties of this graph and establish connections between its graph-theoretical characteristics and the structure of the underlying group.

In this paper, we investigate the relationship between the proposed graph δ_K^g and the g -noncommuting graph (Γ_G^g) . In particular, we prove that the connectivity of δ_K^g implies the connectivity of (Γ_G^g) , whereas the converse does not hold. We then examine the role of the center of the group in the structure of δ_K^g and show how it affects fundamental graph invariants, including the diameter and the girth. In addition, we determine the vertex degrees of δ_K^g . Finally, we apply the proposed construction to the dihedral group to illustrate the obtained theoretical result.

3. Results and Discussion

This section introduces a new class of non-commuting graphs based on properties of derived subgroup, fixed element and the disjoint two elements sets of a group, providing illustrative examples to help highlight the features of this type of graph. We then review a number of key properties associated with the proposed graph δ_K^g .

Definition [2.1] Let G be a finite non-abelian group and let $K = \{A_1, A_2, \dots, A_m\}$ be a family of pairwise disjoint two element subsets. We define the graph which is denoted by δ_K^g as follows, taking $V(\delta_K^g) = K$ a set of vertices and two vertices A_i, A_j are adjacent if and only if for every $a_i \in A_i$ and $a_j \in A_j$ such that $g, g^{-1} \notin \langle a_i, a_j \rangle'$, where $g \in G$

Example [2.2] Assume $G = D_8 = \{r, s : r^4 = e, s^2 = esrs = r^{-1}\}$
 $= e, r, r^2, r^3, s, sr, sr^2, sr^3$, choose $g = r$ and $K = \{A_1 = \{e, r\}, A_2 = \{r^2, r^3\},$
 $A_3 = \{s, rs\}, A_4 = \{r^2s, r^3s\}\}$ then δ_K^r as follows (see figure 1)

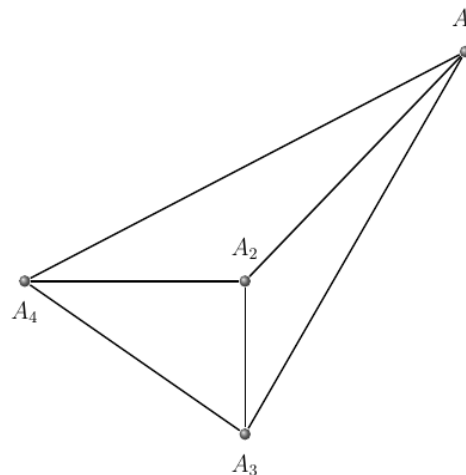


Figure 1: δ_K^r

Remark [2.3] We can see that $|\langle a_i, a_j \rangle'| \leq 4$ for $a_i \in A_i$ and $a_j \in A_j$. If $A_i \sim A_j$ then $g, g^{-1} \notin \langle a_i, a_j \rangle'$ and this indicates that for every $a_i \in A_i$ and $a_j \in A_j$ is adjacent in the graph Γ_G^g . because $[a_i, a_j] \neq g$ and g^{-1} . Also, if $a_i \sim a_j \forall i$ and j , then $A_i \sim A_j$ otherwise $A_i \not\sim A_j$

Lemma [2.4] Let A_i be a sub set of order 2 of a group G then Γ_G^g is connected if δ_K^g is connected, but the opposite of that is not true.

Proof: suppose that δ_K^g is connected graph and let a_i, a_j are two arbitrary vertices of Γ_G^g , then we have the following cases:

cas1: $a_i, a_j \in A_i$, since δ_K^g is connected, so there is A_k such that $A_i \sim A_k$ and this find a path between a_i and a_j

Case2: $a_i \in A_i$ and $a_j \in A_j$, if $A_i \sim A_j$ then by Remark[1.3] $a_i \sim a_j$, if $A_i \not\sim A_j$ and since δ_K^g is connected so there is a path between A_i and A_j as follows

$$A_i \sim A_{v_1} \sim A_{v_2} \sim \dots \sim A_{v_n} \sim A_j$$

and this given that

$$a_i \sim a_{v_1} \sim a_{v_2} \sim \dots \sim a_{v_n} \sim a_j$$

and the proof is complete.

In the reverse direction we can given the following example let $G = D_{10}$ and $g = r^2$, it is clear that Γ_G^g is connected graph, but the graph δ_K^g where

$V(\delta_K^g) = \{\{e, r\}, \{r^2, r^3\}, \{r^4, s\}, \{rs, r^2s\}, \{r^3s, r^4s\}\}$ this graph is not connected because $\{e, r\}$ is not adjacent with any vertices.

Theorem [2.5] Assume G is a group such that $|Z(G)| = 2$ and g non identity element of G then:

(i) δ_K^g is connected graph where it's contain center of G as a vertex.

(ii) $\text{girth}(\delta_K^g) = 3$ where it has a vertex with degree 2.

(iii) $\text{diam}(\delta_K^g) \leq 2$.

Proof (i) Since $Z(G)$ is a vertex and g is non identity element, $Z(G)$ is adjacent with all vertices of δ_K^g

(ii) Let A_i be a vertex of degree 2, thus there exists a vertex A_j adjacent with A_i and we have the vertex $Z(G)$ is adjacent with all elements, so this given the following cycle

$$A_i \sim A_j \sim Z(G) \sim A_i$$

So the result holds.

(iii) If the graph δ_K^g is complete, then every pair of distinct vertices is adjacent. Hence, the distance between any two vertices is equal to one, and therefore $\text{diam}(\delta_K^g) = 1$ Otherwise, let A_i and A_j be two nonadjacent vertices. By the definition of the graph, each vertex is adjacent to the center ($Z(G)$) of the group. Consequently, there exists a path

$$A_i \sim Z(G) \sim A_j$$

of length 2 joining A_i and A_j . Thus, $\text{diam}(\delta_K^g) \leq 2$ Since A_i and A_j are arbitrary nonadjacent vertices, it follows that every pair of vertices is connected by a path of length at most (2). Therefore, the result holds.

Lemma [2.6] Assume A_i is a vertex of δ_K^g then

i) If $g^2 \neq 1$, then $\deg(A_i) = |V(\delta_K^g)| - n|C_{A_j}(a_i)|$ where $n = 1$ if a_i is a conjugate to $a_i g$ or $a_i g^{-1}$ but not both and $n = 2$ if a_i is conjugate to $a_i g$ and $a_i g^{-1}$

(ii) if $g^2 = 1$ and $g \neq 1$, then $\deg(A_i) = |V(\delta_K^g)| - |C_{A_j}(a_i)| - 1$.

(iii) if $a_i g$ and $a_i g^{-1}$ are not conjugate a_i then $\deg(A_i) = |V(\delta_K^g)| - 1$.

Proof: We know that $\deg(A_i) = |V(\delta_K^g)| - |A_j: g \text{ or } g^{-1} \in \langle a_i, a_j \rangle'|$, it is clear that where g or $g^{-1} \in \langle a_i, a_j \rangle'$, then there exist $[a_i, a_j]$ such that $[a_i, a_j] = g$ or g^{-1} and this means that A_i is not adjacent to A_j when $[a_i, a_j] = g \Rightarrow a_i^{a_j} = a_i g$, so every vertex has above condition is not adjacent with A_i . Thus $a_i^{a_j} = a_i^{a_k}$ it's holds when $a_j a_k^{-1} \in C_{A_k}(a_i)$ there for there exist $|C_{A_k}(a_i)|$ elements dose not join A_i so;

i) If a_i and $a_i g$ are conjugate, then $|a_i \in A_k: a_i^{a_j} = a_i g| = |C_{A_k}(a_i)|$, also, if a_i and $a_i g^{-1}$ are conjugate then, $|a_i \in A_k: a_i^{a_j} = a_i g^{-1}| = |C_{A_k}(a_i)|$. Therefore $\deg(A_i) = |V(\delta_K^g)| - n|C_{A_k}(a_i)|$ where $n = 1$ or 2 .

ii) If $g^2 = 1$ and $g \neq e$ then $g = g^{-1}$, so we have a_i conjugate of g and ii) holds.

iii) If $a_i g$ and $a_i g^{-1}$ are not conjugate to a_i then $|C_{A_k}(a_j)| = 0$, thus the proof is complete.

Structural Properties of the Derived Subgroups Associated with the Graph δ_K^g on D_{2n}

The structural properties of the derived subgroups generated by pairs of elements in the dihedral group D_{2n} play a fundamental role in the study of the graph δ_K^g . In this section,

these properties are established and employed to characterize the adjacency relation and derive the main results of the graph.

lemma [3.1][6] Let $D_{2n} = \langle r, s \mid r^n = e, s^2 = e, srs = r^{-1} \rangle$, where n is even, and let $H = \langle x, y \rangle$ for some $x, y \in D_{2n}$. Then $H' \leq D'_{2n} = \langle r^2 \rangle$. Moreover,

$$H' = \begin{cases} \{e\}, & \text{if } H \text{ is abelian,} \\ \langle r^{2d} \rangle, & d \mid \frac{n}{2}. \end{cases}$$

Remark [3,2] Let $G = D_{2n}$, then the number of vertices of the graph δ_K^g is $\frac{n}{2}$.

lemma [3.3][6] Let $x, y \in D_{2n}$.

Then exactly one of the following holds.

- 1) If $x = r^i$, $y = r^j$, then $(\langle x, y \rangle)' = \{e\}$.
- 2) If $x = r^i$, $y = r^j s$, then $(\langle x, y \rangle)' = \langle r^{2d} \rangle$, where $d = \gcd(i, n)$.
- 3) If $x = r^i s$, $y = r^j s$, then $(\langle x, y \rangle)' = \langle r^{2d} \rangle$, where $d = \gcd(i - j, n)$.

corollary[3.4] Let $g \in D_{2n}$. If $g \notin \langle r^2 \rangle$, then $g, g^{-1} \notin (\langle x, y \rangle)'$ for every pair $x, y \in D_{2n}$. So δ_K^g is complete graph.

Proof: Assume that A_i and A_j are two arbitrary vertices of δ_K^g , if A_i is not adjacent with A_j , then g or $g^{-1} \in [a_i, a_j]$, so there exist $[a'_i, a'_j] = g$ or g^{-1} , thus a'_i is not adjacent with a'_j in Γ_C^g . And this contradicts the fact that $\Gamma_{D_{2n}}^g$ is complete graph where $g \notin D'_{2n}$.

Theorem [3.5][Adjacency Criterion]

Let $G = D_{2n}$ and let $K = \{A_1, A_2, \dots, A_n\}$ be a partition of D_{2n} into pairwise disjoint two-element subsets, for a fixed element $g \in D_{2n}$ and for distinct vertices A_i and A_j . Then $A_i \sim A_j$ if and only if $(\langle x, y \rangle)' \cap \{g, g^{-1}\} = \emptyset$ for every $x \in A_i$ and every $y \in A_j$.

Proof Suppose that $A_i \sim A_j$. By the definition of δ_K^g , we have $g, g^{-1} \notin (\langle x, y \rangle)'$ for every $x \in A_i$ and every $y \in A_j$. Hence, $(\langle x, y \rangle)' \cap \{g, g^{-1}\} = \emptyset$ for all such pairs. Conversely, assume that $(\langle x, y \rangle)' \cap \{g, g^{-1}\} = \emptyset$ for every $x \in A_i$ and every $y \in A_j$. Then neither g nor g^{-1} belongs to $(\langle x, y \rangle)'$ for any choice of x and y . Therefore, by the definition of the graph, $A_i \sim A_j$. Thus, $A_i \sim A_j \Leftrightarrow (\langle x, y \rangle)' \cap \{g, g^{-1}\} = \emptyset$ for every $x \in A_i$ and every $y \in A_j$.

4. Conclusion

In this paper, we introduced a new graph associated with a fixed element and the derived subgroup of a group, whose vertex set consists of the two-element subsets of the underlying group. This construction extends the existing framework of algebraic graphs by incorporating binary subsets and the derived subgroup into the adjacency relation.

Several structural properties of the proposed graph were established. In particular, we investigated its relationship with the corresponding g -noncommuting graph and proved that the connectivity of the proposed graph implies the connectivity of the associated g -noncommuting graph, whereas the converse is not valid in general. Furthermore, the influence of the center of the group on the graph structure was analyzed, and fundamental graph invariants, including vertex degrees, diameter, and girth, were determined. The theoretical results were illustrated through their application to the dihedral group D_{2n} .

The proposed construction provides a new approach for investigating finite groups through graph-theoretic techniques. Future work may include extending this graph to other classes of finite groups and studying additional graph invariants and algebraic properties arising from this construction.

REFERENCES

- [1] A. Abdollahi, S. Akbari, and H. R. Maimani, "Non-commuting graph of a group," J. Algebra, vol. 298, pp. 468–492, 2006, doi: 10.1016/j.jalgebra.2005.03.019.
- [2] A. Abdollahi and A. Mohammadi Hassanabadi, "Non-cyclic graph of a group," Commun. Algebra, vol. 35, pp. 2057–2081, 2007, doi: 10.1080/00927870601114511.
- [3] A. H. Alwan and N. K. Tuama, "Nilpotent graph of a semiring," Asia Pac. J. Math., vol. 11, Art. no. 77, 2024.

- [4] A. Iranmanesh and A. Jafarzadeh, "On commuting graphs associated with the symmetric and alternating groups," *J. Algebra Appl.*, vol. 7, no. 1, pp. 129–146, 2008, doi: 10.1142/S0219498808002608.
- [5] A. N. Laabi, A. H. Kamil, and Y. A. Farawi, "Subring in graph theory," *Adv. Nonlinear Var. Inequal.*, vol. 27, pp. 284–287, 2024.
- [6] B. Tolue, A. Erfanian, and A. Jafarzadeh, "A kind of non-commuting graph of finite groups," *J. Sci. Islam. Repub. Iran*, vol. 25, no. 4, pp. 379–384, 2014, doi: 10.22059/jsci.2014.52630.
- [7] D. F. Anderson and P. S. Livingston, "The zero-divisor graph of a commutative ring," *J. Algebra*, vol. 217, pp. 434–447, 1999, doi: 10.1006/jabr.1998.7840.
- [8] D. S. Malik, J. N. Mordeson, and M. K. Sen, *Fundamentals of Abstract Algebra*. New York, NY, USA: McGraw-Hill, 1997.
- [9] F. Aliniaiefard, M. Behboodi, E. Mehdi-Nezhad, and A. M. Rahimi, "The annihilating-ideal graph of a commutative ring with respect to an ideal," *Commun. Algebra*, vol. 42, no. 5, pp. 2269–2284, 2014.
- [10] M. Nasiri, A. Erfanian, Y. A. Farawi, and I. Muchtadi-Alamsyah, "A kind of graph associated to a fixed element and a subgroup of group," *Southeast Asian Bull. Math.*, vol. 44, pp. 813–818, 2020.
- [11] O. K. Khudhair, Y. A. Farawi, and O. G. Khudhair, "Using graphs to depict relationships among elements in various topological spaces," *Nanotechnol. Perceptions*, pp. 800–808, 2024.
- [12] R. Shen, "Intersection graph of subgroups of finite groups," *Czechoslovak Math. J.*, vol. 60, pp. 945–950, 2010, doi: 10.1007/s10587-010-0099-z.
- [13] Y. A. Farawi, "Kind of bipartite graph associated on triple elements of subgroup of a group," *Bol. Soc. Parana. Mat.*, ser. 3, vol. 43, pp. 1–4, 2025, doi: 10.5269/bspm.78346.
- [14] Y. A. Farawi, O. K. Khudhair, and O. G. Khudhair, "Graph associative on disjoint open subsets of topological space," *Bol. Soc. Parana. Mat.*, vol. 44, no. 11, pp. 1–6, 2026, doi: 10.5269/bspm.81560.
- [15] M. Sharma and R. K. Nath, "Relative $\setminus(g\setminus)$ -noncommuting graph of finite groups," *Electronic Journal of Graph Theory and Applications*, vol. 10, no. 1, 2022.