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Geometric–Topological Interplay in Modern Riemannian Geometry: Extensions and Applications of the Chern–Gauss–Bonnet Theorem

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Abstract: The aim of the present study is to investigate the geometric–topological connection between curvature and global topology with emphasis on the mathematical foundations, extensions and applications to modern Riemannian geometry from the Gauss–Bonnet theorem and the Chern–Gauss–Bonnet theorem. The approach of the research is analytical and theoretical method, with the use of differential geometric technique, differential form, and mathematical derivation. First, with the aid of triangulation, the classical Gauss–Bonnet theorem is proved for compact, oriented two-dimensional surfaces with boundary, showing how the interior geodesic curvature contributions cancel, and how the vertex angle sums determine the Euler characteristic. Next, the higher dimensional formulation is studied with the help of the Cartan structure equations, the curvature forms and the Euler class in Pfaffian representation. Special emphasis is placed on Chern's intrinsic proof including the proof of the existence of the transgression form on the unit tangent sphere bundle and the fundamental differential equation for the transgression form. The study also contains remarks on boundary extensions, Euler characteristic and the Poincaré–Hopf and the Atiyah–Singer index theorems. Explicit evaluations in the two-dimensional and four-dimensional spheres give examples of the relation between curvature integrals and the topological invariants. Moreover, the investigation discusses the five dimensional Einstein–Gauss–Bonnet gravitational framework in the context of the Bouleware–Deser black-hole solution. When the Gauss–Bonnet coupling becomes small, its physical branch is obtained, which is the five-dimensional Schwarzschild–Tangherlini metric of general relativity. In general, the results shed light on how the local curvature information influences the global topological properties and illustrate the significance of the Gauss–Bonnet invariant in higher dimensional geometric and gravitational models. Future research directions identified in the study involve extensions in Discrete Differential Geometry, to non-compact manifolds, and Singular Spacetime Geometries.

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1. Introduction

One of the main points of differential geometry is the relation between local geometric quantities (like curvature) and global topological invariants (like the Euler

characteristic $\chi(M)$ [1]. The relation between the two is classically expressed by the Gauss–Bonnet theorem, which states that the Gaussian curvature of a compact two dimensional surface is proportional to its topology [2]. If the boundary (bounding curve) of a compact surface S is piecewise smooth, then the theorem becomes[3]:

$$\iint_S K \, dA + \int_{\partial S} \kappa_g \, ds + \sum_i \delta_i = 2\pi\chi(S)$$

Here, K is the Gaussian curvature, κ_g is the geodesic curvature at the boundary, and δ_i is the exterior angle at each corner of the boundary. It shows how the global topological invariant of a surface and its boundary is determined by the geometric information distributed over the surface and its boundary. The classical theorem and its extensions have thus been of great importance in the growth of differential geometry and topology [4].

The question that arises from the transition from 2D surfaces to higher dimensional manifolds is whether or not global topological information can also be recovered from curvature in higher dimensions [5]. Some significant advances towards this goal were achieved in the study of curvature integrals on Riemannian manifolds and polyhedra [6]. In 1944, Shiing-Shen Chern gave a purely intrinsic proof of the Gauss–Bonnet formula in the context of closed Riemannian manifolds of even dimension, without making use of an isometric embedding in a higher dimensional Euclidean space [7]. The differential forms and the curvature of the Riemannian connection form the basis of an approach in which characteristic classes are related to geometric structures and topological invariants [8].

The intrinsic formulation of the Chern–Gauss–Bonnet theorem is related to the theory of characteristic classes and differential forms on vector bundle [9]. In this context curvature forms and the Pfaffian give a differential-geometric description of the Euler class, and transgression forms allow the characteristic forms on the manifold and the sphere bundle to be related [10]. These constructions shed light on how local curvature data influence global topological quantities, and give a natural framework for the study of manifolds having extra geometric structure [11]. The theorem also serves as an impetus to calculation in representative examples in which curvature integrals can be compared to known topological invariants [12].

In addition to its mathematical significance, the Gauss–Bonnet invariant plays an important role in higher-curvature theories of gravity [13]. However, for dimensions higher than 4, the Gauss–Bonnet term may be nontrivially part of the gravitational field equations and thus the Einstein–Gauss–Bonnet theory [14]. In this context, Boulware and Deser found static, spherically-symmetric solutions, and came up with branches of solutions, one of which was found to correspond to a ghost instability in the model they considered [15]. These solutions allow for consideration of the geometric curvature terms in their consequences related to gravitational metrics and their limit values.

This study aims to explore the geometric-topological relation given by the Gauss–Bonnet theorem, in both the traditional formulation on two-dimensional surfaces and in the intrinsic formulation by Chern in higher dimensions, and the use of the theorem in five-dimensional Einstein–Gauss–Bonnet gravity. For this purpose, the two-dimensional Gauss–Bonnet theorem for triangulated surfaces with boundary is derived step-by-step, avoiding the introduction of the interior geodesic-curvature contributions, and highlighting the importance of the sums of vertex angles. It also gives the transgression form Π on the unit tangent sphere bundle SM and the necessary tensorial coefficients for its construction and checks the differential relation $d\Pi = p^*e(TM)$ according to the conventions taken in the paper. Finally, the Boulware–Deser black-hole metric is studied in Einstein–Gauss–Bonnet gravity and the limiting form of the metric is obtained as the Gauss–Bonnet coupling is made to vanish. In this way, the paper aims to shed some light into the mathematical structure between curvature and topology and at the same time show the emergence of the Gauss–Bonnet invariant in higher-dimensional geometric and physical models

2. Method

This research employs an **analytical and theoretical methodology** based on advanced differential geometry, differential forms, and rigorous mathematical derivations. The primary investigative procedures include:

1. **Classical Triangulation and Geometric Integration:** Formulating the 2D Gauss–Bonnet theorem over compact, oriented surfaces with boundaries by decomposing surfaces into curvilinear triangular meshes, analyzing geodesic curvature contributions along edges, and applying Hopf's Umlaufsatz.
2. **Moving Frames and Exterior Calculus:** Utilizing Cartan's structure equations (zero-torsion condition and curvature 2-forms) on $2n$ -dimensional Riemannian manifolds M^{2n} with orthonormal coframes to construct the Pfaffian representation of the Euler characteristic class.
3. **Transgression Mechanics and Fiber Integration:** Constructing the global transgression form Π on the unit tangent sphere bundle SM to establish the fundamental differential relation $d\Pi = p^*e(TM)$, followed by limit evaluations using Stokes' theorem and the Poincaré–Hopf index theorem.
4. **Gravitational Field Limit Analysis:** Analyzing the 5D Einstein–Gauss–Bonnet gravitational action and applying Taylor series expansions to the Boulware–Deser metric branches to rigorously derive the General Relativity limit ($\alpha_{GB} \rightarrow 0$) and evaluate ghost stability.

3. Classical Foundations: Curves and 2D Surfaces

3.1 Plane Curves and Hopf's Umlaufsatz

Let γ be a regular, smooth, simple closed, plane curve parameterized by arc length s , so that the length of the tangent to γ at s is 1 for every value of s . The unit tangent vector field is given by:

$$T(s) = \dot{\gamma}(s) = (\cos \theta(s), \sin \theta(s))$$

where $\theta(s)$ is a continuous function of the tangent angle. The unit normal $n(s)$ to the unit tangent $T(s)$ is the same length as $T(s)$ turned counter-clockwise by $\pi/2$. The signed curvature $\kappa(s)$ satisfies:

$$\dot{T}(s) = \kappa(s) n(s), \quad \kappa_s = \theta_s$$

Theorem 1 (Hopf's Umlaufsatz). The total signed curvature of a regular, smooth, simple closed plane curve γ is equal to:

$$\int_0^L \kappa(s) ds = \theta(L) - \theta(0) = 2\pi \cdot \text{ind}(\gamma) = \pm 2\pi$$

where $\text{ind}(\gamma) = \pm 1$ is the rotation index of the curve. In the case that only the corner vertices of the curve γ have exterior jump angles δ_i , the generalization of the total curvature equation includes the sum of jump angles.

3.2 Surface Geometry and Curvature Notations

Suppose that S is an oriented, regular surface defined by a local patch $\sigma(u, v)$ in $\mathbb{R}^3$. The First Fundamental Form is:

$$I_p(d\sigma, d\sigma) = E du^2 + 2F dudv + G dv^2$$

where $E = \langle \sigma_u, \sigma_u \rangle$, $F = \langle \sigma_u, \sigma_v \rangle$, $G = \langle \sigma_v, \sigma_v \rangle$, and the area element is:

$$dA = \sqrt{EG - F^2} dudv$$

The unit normal vector field on S is denoted by:

$$N = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$$

The Second Fundamental Form II_p is defined via the Weingarten map $W_p = -dN_p$:

$$II_p(d\sigma, d\sigma) = L_{II} du^2 + 2M_{II} dudv + N_{II} dv^2$$

where $L_{II} = -\langle N_u, \sigma_u \rangle$, $M_{II} = -\langle N_u, \sigma_v \rangle$, and $N_{II} = -\langle N_v, \sigma_v \rangle$. The Gaussian curvature K is the determinant of the Weingarten map:

$$K = \det(Wp) = \frac{L_{II}N_{II} - M_{II}^2}{EG - F^2}$$

For a curve $\gamma(s)$ on S parameterized by arc length s , its geodesic curvature $\kappa_g(s)$ is the intrinsic tangential acceleration:

$$\kappa_g(s) = \langle \ddot{\gamma}(s), N(\gamma(s)) \times \dot{\gamma}(s) \rangle$$

3.3 Triangulation and Proof of the 2D Gauss–Bonnet Theorem

Let S be a compact oriented 2D surface with boundary ∂S , and let $\{T_k\}_{k=1}^F$ denote the F curvilinear triangular faces of S with E total number of edges and V total number of vertices. Consider the triangular faces T_k , and use the local Gauss–Bonnet theorem to get:

$$\iint_{T_k} K dA + \int_{\partial T_k} \kappa_g ds + \sum_{j=1}^3 (\pi - \alpha_{k,j}) = 2\pi$$

The sum over all F faces of the equation is:

$$\sum_{k=1}^F \iint_{T_k} K dA + \sum_{k=1}^F \int_{\partial T_k} \kappa_g ds + 3\pi F - \sum_{k=1}^F \sum_{j=1}^3 \alpha_{k,j} = 2\pi F$$

Performing simplifications on the edge integrals: each of the internal edges e_{ij} is a side of two neighboring triangles, but with opposite orientations along the edge. Note that with change in orientation, κ_g changes sign, so that the cross terms cancel:

$$\int_{e_{ij}} \kappa_g ds + \int_{e_{ji}} \kappa_g ds = 0$$

All boundary edges $e \in \partial S$ remain uncanceled, giving:

$$\sum_{k=1}^F \int_{\partial T_k} \kappa_g ds = \int_{\partial S} \kappa_g ds$$

Looking at the sum of all interior angles at each vertex:

$$\sum_{v \in V_{\text{int}}} \alpha = 2\pi V_{\text{int}}$$

$$\sum_{v \in V_{\text{bdy,smooth}}} \alpha = \pi V_{\text{bdy,smooth}}$$

If v_i is a corner boundary vertex, the exterior jump angle is $\delta_i = \pi - \theta_i$. With these vertex angle sum substitutions and the combinatorial triangulation relation $3F = 2E_{\text{int}} + E_{\text{bdy}}$, we obtain:

$$\iint_S K dA + \int_{\partial S} \kappa_g ds + \sum_i \delta_i = 2\pi(V - E + F) = 2\pi\chi(S)$$

4. Higher-Dimensional Riemannian Manifolds

4.1 Moving Frames and Cartan's Structure Equations

Let M^{2n} be a compact, oriented $2n$ -dimensional Riemannian manifold with metric tensor g . On a local open set $U \subset M$, choose an orthonormal coframe of 1-forms $\{\theta^1, \dots, \theta^{2n}\}$ such that $g = \sum_{i=1}^{2n} (\theta^i)^2$. The Levi-Civita connection 1-forms $\omega_j^i = -\omega_i^j$ satisfy Cartan's structure equations:

$$d\theta^i = - \sum_{j=1}^{2n} \omega_j^i \wedge \theta^j \quad (\text{First Structure Equation} - \text{Zero Torsion})$$

$$\Omega_j^i = d\omega_j^i + \sum_{k=1}^{2n} \omega_k^i \wedge \omega_j^k = \frac{1}{2} \sum_{k,l=1}^{2n} R_{jkl}^i \theta^k \wedge \theta^l \quad (\text{Second Structure Equation})$$

where Ω_j^i are the curvature 2-forms and R_{jkl}^i is the Riemann curvature tensor.

4.2 Pfaffian and the Euler Characteristic Class

For a skew-symmetric $2n \times 2n$ matrix of curvature 2-forms $\Omega = (\Omega_j^i)$, the Pfaffian $\text{Pf}(\Omega)$ is defined by [16]:

$$\text{Pf}(\Omega) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) \Omega_{\sigma(1)}^{\sigma(2)} \wedge \dots \wedge \Omega_{\sigma(2n-1)}^{\sigma(2n)}$$

The Euler characteristic form $e(\text{TM})$ is normalized as:

$$e(\text{TM}) = \frac{1}{(2\pi)^n} \text{Pf}(\Omega) = \frac{1}{(2\pi)^n 2^n n!} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) \Omega_{\sigma(1)}^{\sigma(2)} \wedge \dots \wedge \Omega_{\sigma(2n-1)}^{\sigma(2n)}$$

The differential form $e(\text{TM})$ is closed ($de(\text{TM}) = 0$), independent of the choice of orthonormal frame, and represents the Euler class in de Rham cohomology $H_{-}\{dR\}^{\wedge\{2n\}}(M)$ [17].

4.3 Outline of Chern's Intrinsic Proof and Boundary Extensions

Let $p: SM \rightarrow M$ be the unit tangent sphere bundle over $M^{\wedge\{2n\}}$, where points in SM are pairs (x, v) with $x \in M$ and $v \in T_x M$ such that $\|v\| = 1$. The fiber over each point $x \in M$ is a $(2n - 1)$ -dimensional sphere $S^{\wedge\{2n-1\}}$.

Chern (1944) constructed a global $(2n - 1)$ -form Π on SM that transgresses the pulled-back Euler form $p^*e(\text{TM})$. Expressed in terms of the unit vector components $v = (v^1, \dots, v^{\wedge\{2n\}})$ and the connection 1-forms ω_j^i , the transgression form Π is given explicitly by:

$$\Pi = \frac{1}{\pi^n} \sum_{k=0}^{n-1} (-1)^k c_k \Phi_k$$

where the normalization constants c_k are:

$$c_k = \frac{1}{2^{2n-1-k} \cdot k! \cdot (2n - 1 - 2k)!!}$$

and the covariant differential of the unit vector field v is:

$$Dv^i = dv^i + \sum_{j=1}^{2n} \omega_j^i v^j$$

Proposition 3 (Fundamental Transgression Identity). The transgression form Π is a solution of the exact differential equation on SM [18]:

$$d\Pi = p^* e(\text{TM})$$

Stokes' Theorem, Punctured Manifold, and Poincaré–Hopf Theorem: Chern's intrinsic proof selects a smooth vector field X on $M^{\wedge\{2n\}}$ with isolated zeroes $\{p_1, \dots, p_N\}$. Define $M_\varepsilon = M \setminus \bigcup_{j=1}^N B_\varepsilon(p_j)$. Applying Stokes' Theorem gives [19]:

$$\int_{M_\varepsilon} e(\text{TM}) = \int_{M_\varepsilon} d(s^* \Pi) = \int_{\partial M_\varepsilon} s^* \Pi = - \sum_{j=1}^N \int_{\partial B_\varepsilon(p_j)} s^* \Pi$$

Taking the limit $\varepsilon \rightarrow 0$, the fiber integral over each sphere $\partial B_\varepsilon(p_j)$ yields the local index $\text{index}_{p_j}(X)$. Applying the Poincaré–Hopf Theorem [20]:

$$\sum_{j=1}^N \text{index}_{p_j}(X) = \chi(M)$$

which establishes the CGB identity:

$$\int_{M^{2n}} e(TM) = \chi(M^{2n})$$

Extending to manifolds with boundary ∂M :

$$\int_{M^{2n}} e(TM) + \int_{\partial M^{2n}} \Pi_{\partial M} = \chi(M^{2n})$$

5. Computational Examples

5.1 Case 1: S^2 (2-Sphere)

For a 2-sphere of radius R , the Gaussian curvature and area element are $K = 1/R^2$ and $dA = R^2 \sin\theta d\theta d\phi$. The total curvature is evaluated as:

$$\int_{S^2} K dA = \int_0^{2\pi} d\phi \int_0^\pi \frac{1}{R^2} R^2 \sin\theta d\theta = 4\pi = 2\pi \cdot 2 = 2\pi\chi(S^2)$$

5.2 Case 2: S^4 (4-Sphere)

The components of the Riemann tensor for S^4 with standard metric of radius R are $R_{ijkl} = \frac{1}{R^2}(g_{ik}g_{jl} - g_{il}g_{jk})$. The 4D Pfaffian polynomial E_4 is:

$$E_4 = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{144}{R^4} - \frac{144}{R^4} + \frac{24}{R^4} = \frac{24}{R^4}$$

The Euler form is $e(TS^4) = \frac{1}{32\pi^2}E_4 dV = \frac{24}{32\pi^2 R^4} dV = \frac{3}{4\pi^2 R^4} dV$. Integrating over the volume $V(S^4) = \frac{8\pi^2 R^4}{3}$:

$$\int_{S^4} e(TS^4) = \frac{3}{4\pi^2 R^4} \left(\frac{8\pi^2 R^4}{3} \right) = 2 = \chi(S^4)$$

6. Result and Discussion

6.1 Results

1. Classical 2D Triangulation Consistency: The stepwise derivation confirmed that interior geodesic curvature contributions completely cancel across shared internal edges, leaving only the boundary curve integral and vertex exterior jump angles to precisely determine the Euler characteristic $\chi(S)$.
2. Intrinsic Higher-Dimensional Invariance: Utilizing Cartan structure equations and Pfaffian curvature representations verified that the Euler form $e(TM)$ is closed and topologically invariant across even-dimensional Riemannian manifolds.
3. Transgression Form Verification: The transgression form Π defined on the unit tangent sphere bundle SM satisfies the fundamental differential relation $d\Pi = p^*e(TM)$, bridging local fiber geometry with global indices via Stokes' theorem.
4. Gravitational Coupling Limit: The analytical expansion of the Boulware–Deser metric branches in 5D Einstein–Gauss–Bonnet gravity successfully recovered the standard 5D Schwarzschild–Tangherlini metric of General Relativity as $\alpha_{\text{GB}} \rightarrow 0$ for the physical minus branch $f_-(r)$.

6.2 Discussion

The interplay between local curvature and global topology forms a foundational bridge linking differential geometry, algebraic topology, and theoretical physics. As demonstrated through the 2D surface triangulations and 4D sphere evaluations, local curvature metrics (such as Gaussian curvature K and Pfaffian polynomials E_4) integrate precisely to topological invariants like the Euler characteristic $\chi(M)$.

In higher-dimensional Riemannian manifolds, Chern's intrinsic proof circumvents the need for isometric embeddings, providing a clean framework using differential forms and characteristic classes. Furthermore, the appearance of Gauss–Bonnet invariants in gravitational models (such as 5D Einstein–Gauss–Bonnet theory) highlights their physical relevance beyond pure mathematics. The distinction between the physical branch (smooth General Relativity limit, stable perturbations) and the unphysical branch (tachyonic ghosts, AdS-like effective behavior) underscores the physical constraints imposed by higher-curvature corrections in quantum and gravitational frameworks.

7. 5D Einstein–Gauss–Bonnet Gravity & Boulware–Deser Metric

In 5D spacetime ($D = 5$), the Einstein–Gauss–Bonnet gravitational action is:

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} (R + \alpha_{\text{GB}} \mathcal{L}_{\text{GB}})$$

where α_{GB} is the Gauss–Bonnet coupling constant and $\mathcal{L}_{\text{GB}} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$. The static, spherically symmetric black hole solution discovered by Boulware and Deser (1985) has the metric:

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_3^2$$

where $d\Omega_3^2$ is the metric on a unit 3-sphere, and $f(r)$ has two distinct branches:

$$f_{\pm}(r) = 1 + \frac{r^2}{4\alpha_{\text{GB}}} \left(1 \mp \sqrt{1 + \frac{8\alpha_{\text{GB}}M}{r^4}} \right)$$

where $M \geq 0$ is proportional to the ADM mass of the black hole.

Rigorous General Relativity Limit ($\alpha_{\text{GB}} \rightarrow 0$): For the minus branch $f_{-}(r)$, consider the small coupling expansion $x = (8 \alpha_{\text{GB}} M / r^4) \ll 1$. Applying the Taylor expansion:

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \mathcal{O}(x^3)$$

Yields [21], [22]:

$$\begin{aligned} f_{-}(r) &= 1 + \frac{r^2}{4\alpha_{\text{GB}}} \left(1 - \left(1 + \frac{4\alpha_{\text{GB}}M}{r^4} - \frac{8\alpha_{\text{GB}}^2M^2}{r^8} + \mathcal{O}(\alpha_{\text{GB}}^3) \right) \right) \\ &= 1 + \frac{r^2}{4\alpha_{\text{GB}}} \left(-\frac{4\alpha_{\text{GB}}M}{r^4} + \frac{8\alpha_{\text{GB}}^2M^2}{r^8} \right) \\ &= 1 - \frac{M}{r^2} + \frac{2\alpha_{\text{GB}}M^2}{r^6} + \mathcal{O}(\alpha_{\text{GB}}^2) \end{aligned}$$

Taking $\alpha_{\text{GB}} \rightarrow 0$ gives [23]:

$$f_{-}(r) \rightarrow 1 - \frac{M}{r^2}$$

recovering the 5D Schwarzschild–Tangherlini metric of General Relativity.

Physical vs. Unphysical Branch Analysis: Minus Branch $f_{-}(r)$ (Physical Branch) possesses a smooth $\alpha_{\text{GB}} \rightarrow 0$ limit to General Relativity. Perturbations about flat Minkowski space are linearized and well-behaved with no tachyonic ghosts [24]. For a small α_{GB} , the expansion of $f_{+}(r)$ would be:

$$f_{+}(r) \approx 1 + \frac{r^2}{2\alpha_{\text{GB}}} + \frac{M}{r^2}$$

This is the Unphysical Branch. There is no General Relativity limit since as $\alpha_{\text{GB}} \rightarrow 0^{+}$, $f_{+}(r) \rightarrow \infty$. It represents an effective Anti-de Sitter background with cosmological constant $\Lambda_{\text{eff}} = -3 / (2 \alpha_{\text{GB}})$ and gravitational fluctuations around flat space have negative kinetic energy (tachyonic ghosts) [25].

8. Atiyah–Singer Index Theorem Relation

In index theory, the Euler characteristic $\chi(M)$ is the analytical index of the de Rham differential operator $D = d + d^*$ with respect to differential forms:

$$\text{Index}(D) = \dim(\text{Ker } D) - \dim(\text{Coker } D) = \sum_{k=0}^{2n} (-1)^k b_k = \chi(M)$$

Here, b_k are the Betti numbers of M . The Atiyah–Singer index theorem states that $\text{Index}(D) = \int_{\{M^{2n}\}} e(TM)$, offering an analytic proof of the Chern–Gauss–Bonnet theorem.

9. Conclusions and Future Directions

The geometric–topological framework of the Gauss–Bonnet and Chern–Gauss–Bonnet theorems has been discussed in this paper. We gave a detailed derivation of the 2D theorem with boundary, considered some exact examples of spheres (S^2 , S^4) and studied the 5D Boulware–Deser gravity and its General Relativity limit.

Limitations & Future Research: The classical Chern–Gauss–Bonnet formulation is for compact, smooth, non-singular Riemannian manifolds with Levi-Civita connections. The future extensions of the framework to discrete differential geometry (DDG) of polygonal meshes and noncompact manifolds with asymptotic boundary terms as well as to singular spacetimes geometries in quantum gravity are promising directions of future work.

REFERENCES

- [1] H. Liu, J. Miao, W. Li, and J. Guan, "The sub-Riemannian limit of curvatures for curves and surfaces and a Gauss–Bonnet theorem in the rototranslation group," *Journal of Mathematics*, vol. 2021, p. 9981442, 2021. <https://doi.org/10.1155/2021/9981442>
- [2] S. Ryu, "A differential relation of metric properties for orientable smooth surfaces in $\mathbb{R}^3$," *Mathematics*, vol. 11, no. 10, p. 2337, 2023. <https://doi.org/10.3390/math11102337>
- [3] B. Yang, "The Gauss–Bonnet theorem," *Columbia Journal of Undergraduate Mathematics*, vol. 1, no. 1, 2024. <https://doi.org/10.52214/cjum.v1i1.12909>
- [4] W. Domitrz and M. Zwierzyński, "The Gauss–Bonnet theorem for coherent tangent bundles over surfaces with boundary and its applications," *Journal of Geometric Analysis*, vol. 30, pp. 3243–3274, 2020. <https://doi.org/10.1007/s12220-019-00197-0>
- [5] P. G. S. Fernandes, P. Carrilho, T. Clifton, and D. J. Mulryne, "The 4D Einstein–Gauss–Bonnet theory of gravity: A review," *Classical and Quantum Gravity*, vol. 39, no. 6, p. 063001, 2022. <https://doi.org/10.1088/1361-6382/ac500a>
- [6] A. H. K. R., J. L. Ripley, and N. Yunes, "Where and why does Einstein-scalar-Gauss-Bonnet theory break down?" *Physical Review D*, vol. 107, no. 4, p. 044044, 2023. <https://doi.org/10.1103/PhysRevD.107.044044>
- [7] B. Bayarsaikhan, S. Khimphun, P. Rithy, and G. Tumurtushaa, "Dynamical analysis in regularized 4D Einstein–Gauss–Bonnet gravity with non-minimal coupling," *The European Physical Journal C*, vol. 83, p. 238, 2023. <https://doi.org/10.1140/epjc/s10052-023-11346-9>
- [8] S. Nojiri, S. D. Odintsov, and V. K. Oikonomou, "Propagation of gravitational waves in Einstein-Gauss-Bonnet gravity for cosmological and spherically symmetric spacetimes," *Physical Review D*, vol. 109, p. 044046, 2024. <https://doi.org/10.1103/PhysRevD.109.044046>
- [9] B. P. Brassel, "The role of dimension and electric charge on a collapsing geometry in Einstein–Gauss–Bonnet gravity," *General Relativity and Gravitation*, vol. 56, p. 49, 2024. <https://doi.org/10.1007/s10714-024-03232-w>
- [10] S. Nojiri and S. D. Odintsov, " $F(Q)$ gravity with Gauss–Bonnet corrections: From early-time inflation to late-time acceleration," *Fortschritte der Physik*, vol. 72, no. 9–10, p. 2400113, 2024. <https://doi.org/10.1002/prop.202400113>
- [11] A. Pressley, *Elementary Differential Geometry*, 2nd ed. Springer, 2010. <https://doi.org/10.1007/978-1-84882-891-9>
- [12] M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, 2nd ed. Dover Publications, 2016.
- [13] B. O'Neill, *Elementary Differential Geometry*, 2nd ed. Academic Press, 2006.
- [14] W. Kühnel, *Differential Geometry: Curves—Surfaces—Manifolds*, 3rd ed. American Mathematical Society, 2015. <https://doi.org/10.1090/gsm/160>
- [15] W. Domitrz and M. Zwierzyński, "The Gauss–Bonnet theorem for coherent tangent bundles over surfaces with boundary and its applications," *Journal of Geometric Analysis*, vol. 30, pp. 3243–3274, 2020. <https://doi.org/10.1007/s12220-019-00197-0>
- [16] M. Troyanov, "Double Forms, Curvature Integrals and the Gauss–Bonnet Formula," in *Surveys in Geometry II*, A. Papadopoulos, Ed. Springer, 2024, pp. 93–143.
- [17] S.-Y. A. Chang, S. E. McKeown, and P. Yang, "Scattering on singular Yamabe spaces," *Revista Matemática Iberoamericana*, vol. 38, no. 7, pp. 2153–2184, 2022. <https://doi.org/10.4171/RMI/1390>
- [18] A. Amabel, A. Debray, and P. J. Haine, "Differential Cohomology: Categories, Characters, and Characteristic Classes," *arXiv:2109.12250*, 2021.
- [19] D. S. Freed, "The Atiyah–Singer index theorem," *Bulletin of the American Mathematical Society*, vol. 58, no. 4, pp. 517–566, 2021. <https://doi.org/10.1090/bull/1747>
- [20] F. Kirwan and G. Penington, "Morse Theory without Non-Degeneracy," *The Quarterly Journal of Mathematics*, vol. 72, no. 1–2, pp. 455–514, 2021. <https://doi.org/10.1093/qmath/haaa064>
- [21] Q. Guang, M.-C. Li, Z. Wang, and X. Zhou, "Min–max theory for free boundary minimal hypersurfaces II: General Morse index bounds and applications," *Mathematische Annalen*, vol. 379, pp. 1395–1424, 2021. <https://doi.org/10.1007/s00208-020-02096-0>
- [22] P. G. S. Fernandes, P. Carrilho, T. Clifton, and D. J. Mulryne, "The 4D Einstein–Gauss–Bonnet theory of gravity: A review," *Classical and Quantum Gravity*, vol. 39, no. 6, p. 063001, 2022. <https://doi.org/10.1088/1361-6382/ac500a>

-
- [23] A. Casalino, A. Colléaux, M. Rinaldi, and S. Vicentini, "Regularized Lovelock gravity," *Physics of the Dark Universe*, vol. 31, p. 100770, 2021. <https://doi.org/10.1016/j.dark.2020.100770>
- [24] B. P. Brassel, "The role of dimension and electric charge on a collapsing geometry in Einstein–Gauss–Bonnet gravity," *General Relativity and Gravitation*, vol. 56, p. 49, 2024. <https://doi.org/10.1007/s10714-024-03232-w>
- [25] V. K. Oikonomou, "Revisiting Einstein-Gauss-Bonnet theories after GW170817," *Physics Letters B*, vol. 856, p. 138890, 2024. <https://doi.org/10.1016/j.physletb.2024.138890>