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Article

Preserving Monotonicity of Two-Dimensional Data Using a Three-Parameter Rational Cubic Spline

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Abstract: Preserving the original form of the data whilst inserting new data is a fundamental challenge in many applications, particularly as a number of common insertion methods used in practical applications result in distortions in the graphical representation of the data. This research proposes a fractional spline function as an interpolation function designed to ensure the smoothness of two-dimensional data whilst avoiding certain physical artefacts such as oscillation, as well as providing the ability to control the shape of sub-intervals locally. The proposed function features three positive parameters that can be adjusted manually, thereby allowing control over the tension or curvature of the graphical curve as required. The method relies on the use of the first derivative in its mathematical sense to ensure a smooth transition between each sub-interval of the function's curve, resulting in a final shape that is smooth and free from physical artefacts. The research includes numerical examples and comparisons demonstrating that the proposed method offers high efficiency in maintaining the regularity of the function, whilst also providing flexibility, reliability and computational efficiency.

Keywords: Fractional Spline, Data Interpolation, Shape Preservation

1. Introduction

Data visualisation, particularly when presented in a smooth manner, plays an important role in various fields of knowledge, such as computer-aided geometric design (CAGD) and curve fitting , as data from many scientific fields often exhibit curves with sharp transitions at original points or nodes, or visually undesirable phenomena such as oscillation, or deviate from the intrinsic characteristics of the data, particularly those scattered over wide and uneven intervals.Random data appear in many physical phenomena, engineering problems and scientific applications; for example, the study of tensile strength in mechanical engineering yields random data. Tensile strength can be defined as the maximum force a material can withstand before it breaks [1]. The applied force is usually referred to as stress and is studied in conjunction with the elasticity exhibited by the material; the data resulting from the relationship between stress and elasticity are always monotonic[2].

Scientists have long discussed the issue of approximating and interpolating data to estimate new values that are consistent with the original data, whilst preserving the intrinsic properties of the data- such as positivity, linearity and convexity- and addressing the challenge of continuity and smoothness in the representation of the data . Since the 1960s, researchers have been discussing a number of mathematical techniques and tools for representing data in such a way that it appears smooth whilst retaining its intrinsic properties. The book by [3]. is a cornerstone in the study of spline functions, having compiled and systematised methods of data interpolation and approximation, whilst discussing their advantages and disadvantages[4]. introduced the concept of the constrained spline, which addressed a problem that arose as a side effect of data interpolation, namely the appearance of oscillations in the resulting curves; Although the curves achieved the required smoothness, they suffered from the appearance of inflection points resulting from the interpolation of new data using the traditional spline function (Cubic Polynomial Spline). Through his work, the researcher established algebraic conditions that determine the value of the tension parameter necessary to eliminate these new inflection points . [5] showed further interest in developing the concept of constrained splines through exponential spline functions, and then used the constraint coefficients p_i to control the shape of the curve at the sub-interval level, which made the constraint coefficients more useful in preserving the shape of the data [6][7]. Adopted a piecewise rational spline function (quadratic interpolation) to derive operators linking the control parameter to the derivative and imposing strict constraints on it to ensure the monotonicity of the data was preserved without the need to trial an infinite number of possible values [8].

Extended the quadratic function to a cubic function, as the researchers found that the cubic formulation was more reliable in preserving the convexity of the data and derived lower bounds on the shape parameters that preserve monotonicity [9]. developed sufficient and necessary conditions for preserving positivity by using a cubic spline interpolation that achieves first-order continuity (C^1), Their model lacked the ability to control the intervals of the graphical representation of the data via free parameters; rather, it relied on selecting appropriate derivatives at the knots and then choosing the most suitable curve from among the available options [10]. proposed an improved formulation of the tension concept within a fractional cubic spline function of degree C^2, using a single independent tension parameter for each sub-period, which provided designers with a direct and flexible means of manually controlling the local tension of the curve; increasing the value of the parameter in a given sub-period tightens that section and brings it closer to a straight line without the need to modify the entire curve. [11] proposed a fractional cubic interpolation function of class C^1 with two shape parameters for each sub-interval and derived data-dependent conditions for them, thereby ensuring the preservation of smoothness and convexity of the data. This method provided greater flexibility in controlling the representative curve of the function compared with single-parameter models. [12] imposed conditions on the smoothing parameters to preserve the monotonicity and positivity of the data, whilst also establishing necessary and sufficient conditions to preserve the convexity of the data; they made an important contribution to the field of data imputation by developing an algorithm to automatically select values for the smoothing parameters in a way that preserves the properties of the data[13]. proposed a piecewise fractional cubic function to preserve the positivity of the data; the method relied on imposing constraints to ensure that the curve remained positive, whilst allowing for the addition of new knots in periods where positivity was not achieved.

Although this approach was successful in preserving the shape of the data, the need to include additional knots increases the complexity of curve fitting. In a related context[14]. employed a two-parameter fractional cubic spline, deriving constraints on the values of these two parameters to preserve the monotonic shape of the data; This method achieved the desired objective; however, subjecting the parameters to the constraints necessary for monotonicity limited the designers' freedom to adjust the shape of the curve, which prompted subsequent work to seek models that combine the guarantee of monotonicity with freedom to control the shape. [15] proposed an interpolation function with a cubic numerator and denominator comprising three parameters to maintain the monotonicity of the data; The researchers stipulated that two of the parameters should be positive and free, whilst the third should be subject to strict constraints to maintain regularity. However, the numerical examples provided by the researchers used the two free parameters with equal values in every period, which did not fully demonstrate the effect of their independence on the local control of the curve's shape[16]. utilised a fractional cubic spline function of the first-order continuity class with three parameters to address the problem of maintaining the positivity of two-dimensional data, where one parameter was constrained by data-dependent conditions to ensure positivity, whilst the other two parameters were left free to enable the designer to control the shape of the curve and optimise its visual smoothness. It is noted that the numerical examples considered varying values for the shape parameters across the sub-intervals of the function, which demonstrated in practice the flexibility of the free parameters in locally controlling the shape of the curve. [17] developed a fractional cubic spline function with a cubic numerator and a quadratic denominator, equipped with three shape parameters.

One of the parameters was constrained by conditions sufficient to preserve the shape of the data, whilst the other two were left free to allow local control over the shape of the curve. The method was applied to address the challenge of maintaining non-negativity and data constrained by a straight line succeeded in developing an algorithm that utilises curvature whilst preserving smoothness, thereby producing the smoothest curve that best approximates the original data. The function used comprised two symmetric parameters embedded within a function with a quadratic numerator and a linear denominator; the study aimed to preserve monotonicity, positivity and convexity. [18]. demonstrated that raising the degree of the spline function to the fourth degree can produce a function capable of preserving the properties of monotonicity, convexity and positivity of the data; however, the use of a fourth-degree function adds complexity to the mathematical structure compared with cubic functions.

It is evident from previous studies that interpolation functions designed to preserve the shape and properties of the data have gradually moved towards striking a balance between providing the necessary constraints to preserve the original shape of the data and allowing freedom to control the shape and properties of the curve locally. Nevertheless, restricting the parameters too strictly may limit the flexibility required by designers; With this in mind, the present research proposes a fractional spline function with cubic numerator and denominator, aimed at preserving the regularity of the data whilst providing greater flexibility for local control over the shape of the curve.

First: A cubic rational spline function

Let $(x_i, f_i), i = 0,1,2,\dots,n$ *be a data set such that* $x_0 < x_1 < \cdots < x_n$. *Then a cubic rational spline function can be defined on each subinterval* $I_i = [x_i, x_{i1}]$ *as follows*[3] :

$$S_i(x) = \frac{p_i(\theta}{q_i(\theta)}$$

Where

$$\left.\begin{aligned} S_i(x) &= \frac{q_i(\theta}{p_i(\theta)} \\ p_i &= A_0(1-\theta)^3 + A_1(1-\theta)^2\theta + A_2(1-\theta)\theta^2 + A_3\theta^3 \\ q_i &= u_i(1-\theta)^3 + (u_1 + v_i + w_i)(1-\theta)\theta + v_i\theta^3 \end{aligned}\right\} \quad \dots\dots\dots\dots (1)$$

And where

$$\theta = \frac{x - x_i}{h_i}, \qquad h_i = x_{i+1} - x_i \qquad , i = 0,1,2,\dots,n-1$$

And that function (1) satisfies the first-order continuity condition if

$$S(x_i) = f_i \ , \ S(x_{i+1}) = f_{i+1} \ ,S'(x_i) = d_i \ \ , S'(x_{i+1}) = d_{i+1} \quad \dots\dots (2)$$

where $S(x_i)$ represents the first-order derivative, whilst d_i represents the derivatives at the given data points (nodes); and by substituting the continuity conditions from Equation (2) into Equation (1), we obtain:

$A_0 = u_i f_i$

$A_1 = (u_i + w_i + v_i)f_i + u_i h_i d_i$

$A_2 = (u_i + w_i + v_i)f_{i+1} - v_i h_i d_{i+1}$
$A_3 = v_i f_{i+1}$

Consequently, the numerator can be rewritten as:

$$p_i = u_i f_i (1-\theta)^3 + ((u_i + w_i + v_i)f_i + u_i h_i d_i)(1-\theta)^2\theta + ((u_i + w_i + v_i)f_{i+1} - v_i h_i d_{i+1})(1-\theta)\theta^2 + v_i f_{i+1}\theta^3$$

It is worth noting that if the values of u_i, w_i, v_i are substituted into equation (1), the equation reduces to the standard form of a Hermitian polynomial (Faris et al., 2015).

Secondly: Calculating the derivatives

In this study, the derivatives $d_i$are calculated using a method that preserves the smoothness of the curve, in accordance with the equations below (optimisation is required here):

$$d_i = \begin{cases} 0 & if\ \Delta_{i-1} = 0\ ,\ \Delta_i = 0 \\ \frac{h_i\Delta_{i-1} + h_{i-1}\Delta_i}{h_i + h_{i-1}} & otherwise \end{cases} \qquad i = 2, \dots, n-1$$

$$d_1 = \begin{cases} 0 & if\ \Delta_1 = 0\ \ or\ sgn(d_1^*) \neq sgn(\Delta_1) \\ \Delta_1 + \frac{(\Delta_1 - \Delta_2)h_1}{h_1 + h_2} & otherwise \end{cases}$$

$$d_n = \begin{cases} 0 & if\ \Delta_{i-1} = 0\ \ or\ sgn(d_n) \neq sgn(\Delta_{n-1}) \\ \frac{\Delta_{n-1} + (\Delta_{n-1} - \Delta_{n-2})h_{n-1}}{h_{n-1} + h_{n-2}} & otherwise \end{cases}$$

$$\Delta_i = \frac{f_{i+1} - f_i}{h_i}$$

Thirdly: Maintaining monotonicity of the data using cubic spline interpolation

The condition $f_i < f_{i+1}$, meaning that $[x_0, x_n]$ is a monotonic data set defined over the interval (x_i, f_i) let : the condition that must be satisfied for the function to remain monotonic be $S_i' \geq 0$ for all $x \in [x_i, x_{i+1}]$where $i = 0, n-1$ and since the derivative is given by:

$$S_i' = \frac{q_i(\theta)p_i'(\theta) - q_i'(\theta)p_i(\theta)}{h_i(q_i)^2}$$

In other words, the derivative can be expressed as follows:

$$S_i' = \frac{\sum_{j=0}^{4} C_j (1-\theta)^{4-j}\theta^j}{h_i(q_i)^2}$$

Where:

$$C_0 = u_i^2 d_i$$

$C_1 = 2u_i(u_i + v_i + w_i)\Delta_i - u_i v_i d_{i+1}$
$C_2 = ((u_i + v_i + w_i)^2 + u_i v_i)\Delta_i - u_i v_i(d_i + d_{i+1})$
$C_3 = 2v_i(u_i + v_i + w_i)\Delta_i - u_i v_i d_i$
$C_4 = v_i^2 d_{i+1}$

where $S_i' > 0$ if was $C_j > 0$ where $i = 0,1,2,3,4$

C_0 and C_4 we find $u_i > 0$ و $v_i > 0$ and then

We find C_1 و C_3 from then

$$w_i = \frac{v_i d_{i+1}}{2\Delta_i} - u_i - v_i$$

$$w_i = \frac{u_i d_i}{2\Delta_i} - u_i - v_i$$

And from C_3 we find

$$w_i > \sqrt{u_i v_i \left(\frac{d_i + d_{i+1}}{\Delta_i} - 1\right)} - u_i - v_i$$

Theory 1: A cubic fractional spline function (1) preserves the monotonicity of the graphical representation of monotonic data if:

$$w_i > max\left\{0, \frac{v_i d_{i+1}}{2\Delta_i} - u_i - v_i, \frac{u_i d_i}{2\Delta_i} - u_i - v_i, \sqrt{u_i v_i \left(\frac{d_i + d_{i+1}}{\Delta_i} - 1\right)} - u_i - v_i\ \right\}$$

$$w_i = I_i + \left\{0, \frac{v_i d_{i+1}}{2\Delta_i} - u_i - v_i, \frac{u_i d_i}{2\Delta_i} - u_i - v_i, \sqrt{u_i v_i \left(\frac{d_i + d_{i+1}}{\Delta_i} - 1\right)} - u_i - v_i \right\}$$

Where: $I_i > 0$

Fourth: Numerical examples and proofs

Example 1

Table 1 shows a ordinal data set [11] , Figure 1 shows the data without the use of any smoothing method, whilst Figure 2 shows the data using the cubic Hermite, interpolation method; the loss of periodicity in the interval [3, 9] is clearly evident

Table 1. Original Ordinal Data Set

X	0	2	3	9	11
Y	0	4	7	9	13

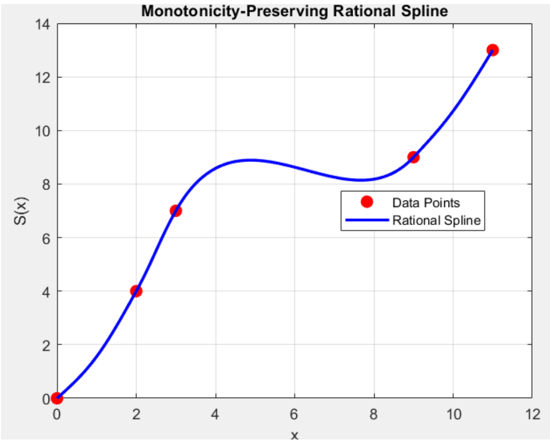

Figure 1. Original Data without Smoothing

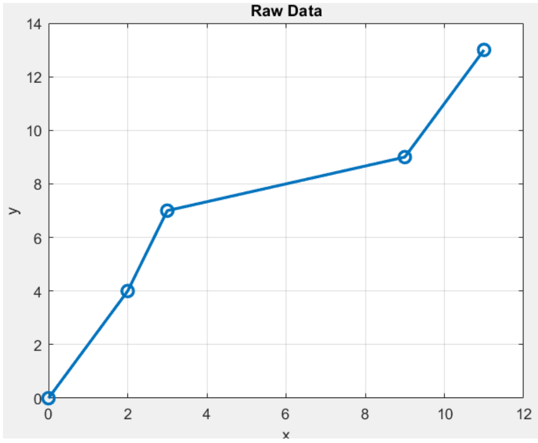

Figure 2. Data Interpolation Using the Cubic Hermite Method

Figure 3 shows a data plot using the function proposed in this study, assuming that the free parameters u_i and v_i are positive, as suggested by mathematical theory, whilst standardising the values of the tension parameter W_i across all sub-periods whilst Figure 4 shows the data representation after maximising the value of the tension parameter W_i in the third period, where we assumed that $I_i = 5$

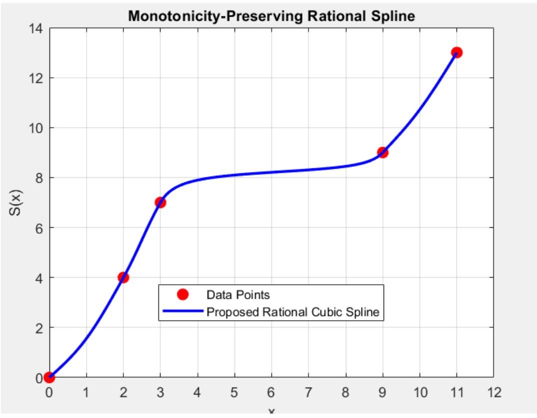

Figure 3. Data Representation Using the Proposed Function

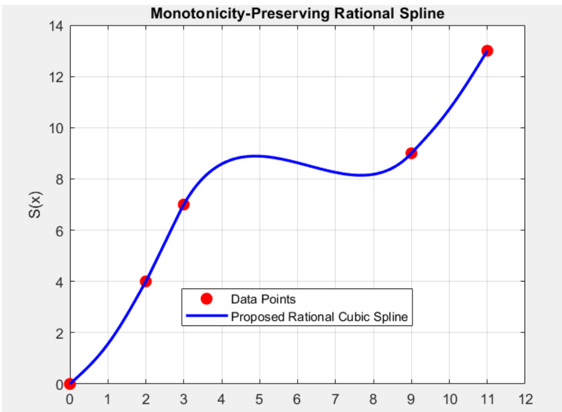

Figure 4. Data Representation with Maximized Tension Parameter in the Third Period

2. Conclusions

a. It was observed that data interpolation using the standard Hermite method does not preserve monotonicity, despite its ability to produce a smooth curve, as local dips may appear in some sub-intervals.

b. The method proposed here demonstrated an ability to preserve the regularity of the data through the constraints imposed on the shape parameters; however, the numerical experiment showed that the initial choice of parameters was insufficient to preserve regularity in the third sub-period. The freedom to select different parameter values for each sub-period allowed this issue to be addressed locally; adjusting the parameter values for that sub-period in accordance with the proposed conditions restored smoothness without the need to alter the parameters for the remaining sub-periods.

c. The results showed that the choice of derivatives at the nodes is an important factor in the behaviour of the resulting curve; an appropriate estimation of the derivatives, consistent with the trend of the data, helps to minimise undesirable deviations, supports the stability of the curve and maintains its monotonic trend.

Recommendations

Based on the above results, it is recommended that more efficient strategies for selecting the optimal values of the free-form parameters be investigated, rather than relying on manual selection, whilst also examining the impact of different derivative estimation methods on the quality and stability of the curve. The proposed method could also be extended in future to address other shape-preservation properties, such as convexity and the preservation of data above a straight line, and to investigate the possibility of raising the continuity class from C^1 to C^2.

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