CENTRAL ASIAN JOURNAL OF MATHEMATICAL THEORY AND COMPUTER SCIENCES
https://cajmtcs.centralasianstudies.org/index.php/CAJMTCS
Volume: 07 Issue: 04 | October 2026 ISSN: 2660-5309

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Article

The Co-e-Filters Among $C^*(X)$

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Abstract: Here we investigate the Co-ϵ-filter sometimes we say Co-e-filter (filters of Cozero e-set) and e-ideal association of $C^*(X)$ for real-valued bounded continuous functions on Tychonoff space X, the essential concept of the study is Coz-filters of bounded continuous functions herein denoted by Co-e-filters for non-compact spaces. We establish the relationship among Co-ϵ-ultrafilters and the minimal prime ideals of $C^*(X)$ providing a direct analogue to the known relationship between e-ultrafilters and maximal ideals, the central result demonstrates a precise correspondence between these co-e-ultrafilters and the minimal prime z-e-ideals of $C^*(X)$, thereby mirroring the established association in $C(X)$ within the bounded context.

Keywords: Co-e-filter, ϵ-set, prime z-ϵ-ideal, Co-e-ultrafilter, $C^*(X)$.

Citation: Rasheed, M. A., & Ghadermazi, M. The Co-e-Filters Among $C^*(X)$. Central Asian Journal of Mathematical Theory and Computer Sciences 2026, 7(4), 112-123.

Received: 10th May 2026
Revised: 12th Jun 2026
Accepted: 25th Jul 2026
Published: 24th Aug 2026

1. Introduction

Among this work X is completely regular (Tychonoff) space and $C^*(X)$ ring of bounded continuous real-valued functions i.e., $C^*(X)$ consist those functions in $C(X)$ s.t. $|sup_{x\in X}|f(x)|<\infty$, thus, $C^*(X)\subseteq C(X)$. It is well known that there is a natural correspondence Z between ideals of $C(X)$ and z-filters on X as described in [1], in fact many theorems that hold for $C(X)$ are no longer valid with C^*. However, the E correspondence between certain types of z-filters on X and ideals in $C^*(X)$ produce results similar to those for $C(X)$ see 2L in [1].

Moreover, the analogue of Theorem 2.3(a) in [1] does not generally hold in $C^*(X)$, e.g., $Z\,[J]$ of an ideal $J\subseteq C^*(X)$ satisfies two of the three axioms of a z-filter, it may fail the first axiom, see example 2.4, J in $C^*(\mathbb{N})$ in [1], which clarify the problem. The coz-filters and prime ideal relationship of $C(X)$ explored in [2], as Coz-ultrafilters connection with minimal prima ideals established. More on Coz-ultrafilters you can see in [3]. The z-filters-ideal relationship and some remarks on them for certain rings of $C(X)$ explored in [4], and for more on prime z-filters on completely regular spaces refer to [5].[6], [7] and [8] investigates the relationships $\mathcal{Z}_A$ and $\mathfrak{Z}_A$, between ideals in $C^*(X)$ to z-filters and ideals in $C(X)$ to z-filters on X, exploring the shared properties of $C^*(X)$ and $C(X)$ by intermediate rings. [9] studied prime and maximal ideals among subrings that include $C^*(X)$. Many problems about ideals in $C(X)$ explored in [10]. Refer to [11] and [12] for multiplicative sets. For prime ideal framework in $C^*(X)$ see [13] and refer to [14] and [15] for on prime ideal with z-ideals in $C^*(X)$, and some results on ultrafilters compared in [16].

Inspiring by previous, we investigate the association between ideals in $C^*(X)$ and Coz-filters on X, focusing on the relationship between prime ideals and cozero ϵ-filters (recall Co-e-filters) of $C^*(X)$. We give a characterization (Definition 2.6-2.7 and Theorem 3.7-4.6) of the correspondence E and CoE for the ring $C^*(X)$. A key contribution is the establishment of a correspondence between co-e-ultrafilters (analogous to Coz-ultrafilters) and minimal prime ideals of $C^*(X)$, mirroring the known correspondence between e-ultrafilters and maximal ideals.

In Ch-2 in [1] the Z correspondence among z-filters and ideals in $C(X)$ is established. In the same chapter E correspondence which is Z-analogue of C^*-version also studied, both correspondences are valuable tools for our study, they may not have same impact according to the C and C^* structure, but they still have some connections as shows in 2.14, $Z^{\leftarrow}[\mathcal{F}] \cap C^* = I$ is an ideal in $C^*(X)$. Also $C^*(X)$ forms a commutative ring with unity, and it shares some structural similarities with $C(X)$ but some concepts differ, e.g., zero divisors and units, an f in $C(X)$ might be a zero divisor but not in $C^*(X)$ (or vice versa). It is obvious in $C^*(X)$, a function f is a unit iff $|f| \geq r$ for some $r > 0$. (i.e., $inf_{x\in X}|f(x)| > 0$), (e.g., $f(x) = 1$ is invertible but $f(x) = \frac{1}{(1+x^2)}$ is not, in $C^*(\mathbb{R})$) (since it approaches zero). But in $C(X)$ a function f is invertible iff $f\ (x) \neq 0, \forall x \in X$, (e.g., $f\ (x) = e^x$ is invertible in $C(\mathbb{R})$, but $f^{-1}(x) = e^{-x}$ is unbounded), key differences arise due to the boundedness condition imposed on $C^*(X)$.

2. Co-e-filter

On non-compact spaces $C^*(X)$ is proper subring of $C(X)$, simply this means the results held in $C(X)$ may hold again but with extra conditions for C^* during non-compactness and properness, informations and results mentioned in 2L [1], are basis in this work, the authors introduce the concepts of ϵ-filters and ϵ-ideals in an effort to preserve correspondence of $C^*(X)$, this refinement is necessary due to the failure of many results, valid in $C(X)$, to hold in $C^*(X)$.

Zero set as established in [1] are of the form $E_\varepsilon(f) = \{x \in X: |f(x)| \leq \varepsilon\}$ for f in C^* with $\varepsilon > 0$ and they are closed in X, on other hand every zero set $Z(g)$ can be transformed in the form $E_\varepsilon(f)$ for some $f \in C^*(X)$ and $\varepsilon > 0$, absolutely this works since ϵ-condition guarantee's that z-filter is ruled by bounded functions, so the ideal I is determined by the behavior of its functions at infinity via $E_\varepsilon(f)$.

3.1 Proposition. for any f and g in C^* with $\epsilon, \varepsilon > 0$ then the following holds for their e-set.

1. $E_{\min(\epsilon^2,\varepsilon^2)}(f^2 + g^2) \subseteq E_\epsilon(f) \cap E_\varepsilon(g)$
2. $E_\epsilon(f) \cap E_\varepsilon(g) \subseteq E_{\epsilon^2+\varepsilon^2}(f^2 + g^2)$

Proof.

1. Consider x be in $E_{\min(\epsilon^2,\varepsilon^2)}(f^2 + g^2)$ which gives $f^2(x) + g^2(x) \leq min(\epsilon^2, \varepsilon^2)$ but $f^2(x)$ and $g^2(x)$ are greater than or equal to zero then each one can be bound independently, for f, $f^2(x) \leq f^2(x) + g^2(x) \leq min(\epsilon^2, \varepsilon^2) \leq \epsilon^2 \Longrightarrow |f(x)| \leq \epsilon \Longrightarrow x \in E_\epsilon(f)$, $g^2(x)$ same pattern $g^2(x) \leq f^2(x) + g^2(x) \leq min(\epsilon^2, \varepsilon^2) \leq \varepsilon^2 \Longrightarrow |g(x)| \leq \varepsilon \Longrightarrow x \in E_\varepsilon(g)$, since x belongs to both sets, so, $x \in E_\epsilon(f) \cap E_\varepsilon(g)$.
2. Let x belong to $E_\epsilon(f) \cap E_\varepsilon(g)$, so, $|f(x)|$ less than or equal to ϵ and $|g(x)|$ less than or equal to ε, then squaring both sides of the inequalities still remain the same, thus $f^2(x) + g^2(x) \leq \epsilon^2 + \varepsilon^2$, so $x \in E_{\epsilon^2+\varepsilon^2}\ (f^2 + g^2)$.

2.2 Proposition. In Tychonoff space X every coz-set of unbounded continuous function $f \in C(X)$ coincides with the cozero set of some bounded continuous function $g \in C^*(X)$. For a given $f \in C(X)$, define $g: X \to \mathbb{R}$ by, $g(x) = \frac{f(x)}{1+|f(x)|}$, this function is well-defined; with these properties, the denominator $1 + |f(x)|$ never vanishes, and f is continuous, so g is continuous, $|g(x)| = |f(x)|$, and $1 + |f(x)| < 1$ for all $x \in X$, hence $g \in C^*(X)$. If $f(x) \neq 0$, then $g(x) \neq 0$ (because $f(x) = 0 \Leftrightarrow g(x) = 0$), thus, $coz(g) = coz(f)$ (see 1.15 in [1]).

The explored relationship between z-filters and ideals in [1] establishes a related fundamental theorem to our study which serves as a foundational tool, empowerment a deeper exploration of the relationship between these structures, next we explore an important theorem which is use full for our work.

2.3 Theorem (see [1]2L-5). ϵ-ideals in $C^*(X)$ and ϵ-filters correspondence on X.

(a) Let I be ϵ-ideal in $C^*(X)$ then $E(I) = \{E_\epsilon(f): f \in I, \epsilon > 0\}$ is ϵ-filter on X.

(b) Let $\mathcal{F}$ be ϵ-filter on X then $E^-(\mathcal{F}) = \{f \in C^*: E_\epsilon(f) \in \mathcal{F}\ for\ all\ \epsilon > 0\}$ is ϵ-ideal in $C^*(X)$.

Proof.

(a) Every $Z = E_\epsilon(f) \in E(I)$ and $\varepsilon > 0$, $E_\varepsilon(f) \in E(I)$, this follows because $f \in I$ and $E_\varepsilon(f)$ is a zero-set containing $E_\epsilon(f)$ when $\varepsilon \geq \epsilon$. Now let $f \in I$, $\epsilon > 0$, and $E_\epsilon(f)$, if $E_\epsilon(f) = \emptyset$, so, $|f(x)| > \epsilon\ for\ all\ x \in X$ that is f is bounded away from zero, gives f is unit in $C^*(X)$, and this contradicts I being a proper, thus $\emptyset \notin E(I)$. Let the zero-sets $E_\epsilon(f), E_\varepsilon(g)$ be in $E(I)$ with $f, g \in I$, consider $h = f^2 + g^2 \in I$ (since I is ideal), for any $x \in E_{\min\left(\epsilon^2, \varepsilon^2\right)}(h)$, we have $f^2(x) + g^2(x) \leq \min(\epsilon^2, \varepsilon^2)$, this implies both $|f(x)| \leq \epsilon\ and\ |g(x)| \leq \varepsilon$ inmplies $E_{\min\left(\epsilon^2, \varepsilon^2\right)}(h) \subseteq E(I)$, by proposition 2.1, $E_{\min\left(\epsilon^2, \varepsilon^2\right)}(h) \subseteq E_\varepsilon(f) \cap E_\varepsilon(g) \in E(I)$. And let $E_\epsilon(f) \in E(I)$ and $Z \in Z(X)$ with $E_\epsilon(f) \subseteq Z$, by Proposition 2.2, $Z = E_\varepsilon(g)$ for some $g \in C^*(X)$, construct a function $h \in C^*(X)$ that equals 1 on $E_\epsilon(f)$ and $g + \epsilon/f$ outside. Then $Z = E_\varepsilon(g) = E_\epsilon(fh) \in E(I)$ since $fh \in I$.

(b) Let $f, g \in E^-(\mathcal{F})$, for any $\epsilon > 0$, $E_{\epsilon/2}(f), E_{\epsilon/2}(g) \in \mathcal{F}$, then $E_{\epsilon/2}(f) \cap E_{\epsilon/2}(g) \subseteq E_\epsilon(f - g)$, but $\mathcal{F}$ is filter, $E_\epsilon(f - g) \in \mathcal{F}$. Thus $f - g \in E^-(\mathcal{F})$. Let f be in $E^-(\mathcal{F})$ and $h \in C^*(X)$ then for any $\epsilon > 0$, $E_{\epsilon/\|h\|}(f) \in \mathcal{F}$ sothat $E_{\epsilon/\|h\|}(f) \subseteq E_\epsilon(fh)$, thus $E_\epsilon(fh) \in \mathcal{F}$, thus $fh \in E^-(\mathcal{F})$.

Consider f be in $C^*(X)$ and $E_\epsilon(f) \in E(E^-(\mathcal{F}))$ for all $\epsilon > 0$, so that for each $\epsilon > 0$, $\exists g \in E^-(\mathcal{F}), \varepsilon > 0$ with $E_\varepsilon(g) \subseteq E_\epsilon(f)$ this $f \in E^-(\mathcal{F})$ by filter properties, thus $E^-(E(E^-(\mathcal{F}))) = E^-(\mathcal{F})$.

As clarified in above theorem, for $f \in C^*(X)$ and $\epsilon > 0$, the zero ϵ-set of f is $E_\epsilon(f) = \{x \in X: |f(x)| \leq \epsilon\}$, each $E_\epsilon(f)$ is zero-ϵ-set in X, for $I \subseteq C^*(X)$.

$$E(I) = \bigcup_{\epsilon > 0} E_\epsilon[I]$$

For a z-filter $\mathcal{F}$ on X.

$$E^\leftarrow(\mathcal{F}) = \bigcap_{\epsilon > 0} E_\epsilon^\leftarrow[\mathcal{F}]$$

The mapping E and $E^\leftarrow$ establish the correspondence between ideals in $C^*(X)$ and collection of zero sets, E maps ideals in $C^*(X)$ to ϵ-filters, $E^\leftarrow$ maps z-filters to ideals in $C^*(X)$, when $I \subseteq J \Rightarrow E(I) \subseteq E(J)$ and $\mathcal{F} \subseteq \mathcal{G} \Rightarrow E^\leftarrow(\mathcal{F}) \subseteq E^\leftarrow(\mathcal{G})$, so that $E, E^\leftarrow$ are inclusion-preserving (see [1] 2L, 5-7).

An ideal $I \subseteq C^*(X)$ is called an ϵ-ideal if $E^\leftarrow(E(I)) = I$, equivalently, I is an ϵ-ideal if and only if it satisfies $E_\epsilon(f) \in E(I)\ for\ all\ \epsilon > 0$, then $f \in I$. In other words, I contains every function whose ϵ-zero sets all belong to $E(I)$, thus ϵ-ideal is special kinds of ideal, and their intersections remain ϵ-ideal, however, not every ideal is ϵ-ideal, in general, $I \subseteq E^\leftarrow(E(I))$, with equality precisely when I is an ϵ-ideal, the z-filter $\mathcal{F}$ is called ϵ-filter if $E(E^\leftarrow(\mathcal{F})) = \mathcal{F}$, so ϵ-filters form a distinguished subclass of z-filters, a z-filter $\mathcal{F}$ is an ϵ-filter if and only if for every $Z \in \mathcal{F}$, there exist $f \in C^*(X)$ and $\epsilon > 0$ such that $Z = E_\epsilon(f)$ and $E_\delta(f) \in \mathcal{F}$ for all $\delta > 0$, the ϵ-condition ensures that $\mathcal{F}$ is generated by bounded functions, means ϵ-filters as special kinds of z-filters, in particular, every ϵ-filter is generated by bounded functions.

Moreover, every ideal $I \subseteq C^*$ gives rise to an ϵ-filter $E(I)$, and vice versa, if $\mathcal{F}$ is ϵ-filter then $\mathcal{F} = E(I)$ or $I = E^{\leftarrow}(\mathcal{F})$. Conversely, $E(I)$ is ϵ-filter for every ideal $I \subseteq C^*(X)$, remember that not every z-filter is an ϵ-filter; in general, $E(E^{\leftarrow}(\mathcal{F})) \subseteq \mathcal{F}$, with equality precisely when $\mathcal{F}$ is an ϵ-filter, we clarify more in the next example.

The condition $E_\epsilon(f) \in \mathcal{F}\ for\ all\ \epsilon > 0$ replaces the simpler $Z(f) \in \mathcal{F}$. The theory is more involved because C^* lacks the flexibility of C (e.g., unbounded functions) (for more refer to [1]2L-1-3).

2.4 Example. Consider the z-filter consisting of all cofinite sets in $\mathbb{N}$ and let $\mathcal{F}$ be the family of all cofinite subsets of $\mathbb{N}$. In the discrete topology on $\mathbb{N}$. Every subset is a zero set (since $\mathbb{N}$ is completely regular and discrete), thus $\mathcal{F}$ is a z-filter since $\mathcal{F}$ is satisfies the axioms of z-filter (since the intersection of two cofinite sets is cofinite, and a superset of a cofinite set is cofinite), but $\mathcal{F}$ is not ϵ-filter, consider a countable family of cofinite sets $\{Z_n\}$ where $Z_n = \mathbb{N}\backslash\{n\}$, each Z_n is cofinite (its complement is $\{n\}$ which is finite), the intersection of

$$\bigcap Z_n = \bigcap \mathbb{N}\backslash\{n\} = \mathbb{N}\backslash\bigcup_n\{n\} = \mathbb{N}\backslash\mathbb{N} = \emptyset$$

But this is contradiction to filter axioms, thus $\mathcal{F}$ is not e-filter.

Furthermore in [17], the authors discussed about E-regularity, they showed that $f \in A(X)$ is E-regular in $A(X)$ (i.e., $\exists g \in A(X)$ with $fg\ |_E = 1$), so on the set E, f is nowhere zero and $g = \frac{1}{f}$ (is multiplicative inverse of f) belongs to $A(X)$, take $h = \max\{c, f\}$ which equals f on E and is bounded below by c, then h^{-1} is bounded (by $\frac{1}{c}$) and continuous, hence in $C^*(X) \subseteq A(X)$, and serves as the required inverse on E.

2.5 Lemma. Analogue to Lemma1(c) in [17], if $f(x) \geq c > 0$ for all $x \in E$, in $C^*(X)$, specifically, f is E-regular in $C^*(X) \Leftrightarrow \inf_E|f| > 0$.

Proof. $\Rightarrow$ suppose $\inf_E|f| = m > 0$, then $|f| \geq m$ on E, so by Lemma 1(c) in [17] with $c = m$, f is E-regular.

$\Leftarrow$ suppose f be E-regular, then there exists $g \in C^*(X)$ with $fg\ |_E = 1$, then $|f| = \frac{1}{|g|}$ on E, since g is bounded, $|\ g\ | \leq M$ this implies $|\ f\ | \geq 1/M > 0$ on E. Hence the infimum is positive.

This is strictly stronger than invertibility condition in $C(X)$, where f need only be nonzero pointwise (i.e., $\mathrm{coz}(f) = X$), the classic counterexample, $f(x) = \frac{1}{1+x^2}$ on $\mathbb{R}$ is nonzero everywhere but has infimum 0, so it is not X-regular in $C^*(\mathbb{R})$. So, by using the E-regularity, the authors defined, for any $f \in A(X)$,

$$\mathcal{Z}(f) = \{\, E \in \mathcal{Z}[X]: f \text{ is } E^c\text{-regular} \,\}$$

recall, $\mathcal{Z}[X]$ is family of all zero-sets in X, and E is zero e-set, so, $E^c = X \setminus E$ is cozero set, i.e. $\mathcal{Z}(f)$ only collects those zero sets whose complement are set where f becomes invertible in $A(X)$, ([17], p, 764).

Now, by combine these facts specifically for $A(X) = C^*(X)$ it to derive,

$$\mathcal{E}_{C^*}(f) = \{E \in \mathcal{E}[X]: \inf_{E^c}|f| > \epsilon, \epsilon > 0\}$$

$E[B] = \bigcup_{f\in B} E(f)$ for $B \in C^*(X)$ we remember the z-filter property, which is that a nonempty family $\mathcal{F}$ of zero e-sets in X s.t., $\mathcal{F} = \mathcal{H} \cap \mathcal{E}[X]$ when $\mathcal{H}$ is filter on X.

According to the above notes, we can investigate the relation between prime ideals in $C^*(X)$ and coz-filters on X, since in z-filter case the authors delved into concepts, ϵ-filters and ϵ-ideals in $C^*(X)$ and coz-filter is the dual of z-filter, we also do this to ensure the boundness property (on noncompact spaces). First let define the co-ϵ-set (dual of ϵ-set), then define the Co-ϵ-filter, and investigate the relationships between ϵ-ideals and co-ϵ-filters on $C^*(X)$.

2.6 **Definition.** For a function $f \in C^*(X)$ and $\epsilon > 0$ we define, $CoE_\epsilon(f) = E_\epsilon(f)^c = X \backslash E_\epsilon(f) = \{x \in X: |f(x)| > \epsilon\}$, every such set is cozero set in $C^*(X)$, here we call it co-e-set (or co-ϵ-set).

2.7 **Definition.** Let f be in $C^*(X)$ we define the Co-ϵ-filter of f such that.

$$CoE_{C^*}(f) = \left\{V \in \mathrm{CoE}(X): \epsilon > 0, \inf_V |f| > \epsilon\right\}$$

Now, every cozero set $V \in \mathrm{CoE}(X)$ is the complement of a zero set $E = X \setminus V \in \mathcal{E}[X]$ put $V = E^c$, dualizing $\mathcal{E}_{C^*}(f)$ directly gives $CoE_{C^*}(f)$. In another word, Co-e-filter on X (for non-compact spaces) is the collection $\mathcal{F}$ of co-ϵ-sets of functions in $C^*(X)$, if it satisfies the coz-filters condition and with additional ϵ-property, for every $A \in \mathcal{F}$, there exists $f \in C^*(X)$ and $\epsilon > 0$ such that $CoE_\epsilon(f) \subseteq A$ and $CoE_\delta(f) \in \mathcal{F}$ for all $\delta > 0$.

The e-condition ensures that the coz-filter $\mathcal{F}$ would be generated by bounded functions.

An ideal $I \subseteq C^*$ is ϵ-ideal if $CoE^-(CoE(I^c)) = I^c$, and a Coz-filter $\mathcal{F}$ is a Co-ϵ-filter if $CoE(CoE^-(\mathcal{F})) = \mathcal{F}$, but not every ideal I satisfies $CoE^{\leftarrow}(CoE(I)) = I$, and not every Coz-filter $\mathcal{F}$ satisfies $CoE(CoE^{\leftarrow}(\mathcal{F})) = \mathcal{F}$ (comparing to [1] 2L-1,3), this ensures that e-ideals and co-e-filters are proper subclasses of ideals and Coz-filters, $\mathcal{F} \supseteq CoE(CoE^-(\mathcal{F}))$, $I^c \subseteq CoE^-(CoE(I^c))$. And the condition $CoE_\epsilon(f) \in \mathcal{F}\ \forall\ \epsilon > 0$ replaces $coz(f) \in \mathcal{F}$.

The determined results in [1] 2L are well known, but ϵ-ideals $\rightleftarrows$ Co-ϵ-filters correspondence in $C^*(X)$ on noncompact paces still not examinate yet as in C. Next we try out to explore this connection in detail, the exploration will elucidate the differences between ϵ-filters and co-ϵ-filters. Note that in this correspondence ideals in C^* must be prime.

2.8 **Proposition.** Let f, g be in C^*, and $\epsilon, \varepsilon > 0$ then for their co-e-set the following holds.

(1) $\mathrm{CoE}_{\epsilon\varepsilon}(fg) \supseteq \mathrm{CoE}_\epsilon(f) \cap \mathrm{CoE}_\varepsilon(g)$

(2) $\mathrm{CoE}_{\min\left(\epsilon^2, \varepsilon^2\right)}(f^2 + g^2) \supseteq \mathrm{CoE}_\epsilon(f) \cup \mathrm{CoE}_\varepsilon(g)$

Proof.

(1) Let $x \in CoE_\epsilon(f) \cap CoE_\varepsilon(g)$ which give $|f(x)| > \epsilon > 0$ and $|g(x)| > \varepsilon > 0$, so, $|f(x)g(x)| = |f(x)||g(x)| > \epsilon\varepsilon$, so that, $x \in CoE_{\epsilon\varepsilon}(fg)$.

(2) Let $x \in CoE_\epsilon(f) \cup CoE_\varepsilon(g)$ this means $|f(x)| > \epsilon$ or $|g(x)| > \varepsilon$, squaring the functions implies $f^2(x) > \epsilon^2$ or $g^2(x) > \varepsilon^2$ since $f^2(x)$ and $g^2(x)$ are non-negative real numbers so $f^2(x) + g^2(x) \geq f^2(x)$ and $f^2(x) + g^2(x) \geq g^2(x)$. Now, if $|f(x)| > \epsilon$, then $f^2(x) + g^2(x) > \epsilon^2 \geq min(\epsilon^2, \varepsilon^2)$, if $|g(x)| > \varepsilon$, then $f^2(x) + g^2(x) > \varepsilon^2 \geq min(\epsilon^2, \varepsilon^2)$, in either case, the sum of $f^2(x)$ and $g^2(x)$ is guaranteed to be greater than minimum of the two squared bounds, thus, we have $x \in CoE_{min(\epsilon^2, \varepsilon^2)}(f^2 + g^2)$.

2.9 **Lemma.** Whenever P be prime in $C^*(X)$ with f, g belonging to $C^*(X)$ such that $f^2 + g^2 \in P$, it follows that f belong to P as well g. it is important to note that, this property is true for any prime ideal P in C^*, because the prime ideals in $C^*(X)$ said to be absolutely convex, i.e., if $0 \leq |a| \leq |b|$ for $b \in P$ we get $a \in P$, because $f^2 \leq f^2 + g^2$ same way g^2 do, moreover when $f^2 + g^2 \in P$ implies $f^2 \in P$ and $g^2 \in P$, consequently, since P is prime, if $f^2 = f \cdot f \in P \Rightarrow f \in P$, and similarly $g \in P$, for more see [1] 2D-2 and 5D).

2.10 **Lemma.** (see [1] theorem 5.4). e-ideals in $C^*(X)$, all maximal ideals (more generally, all prime ideals) are absolutely convex.

2.11 **Lemma.** (Absolute convexity). Every prime ideal in $C^*(X)$ is absolutely convex, if $|h| \leq |k|$ pointwise and $k \in P$, then $h \in P$.

In the following theorem, we will explore the goal of this section.

2.12 **Theorem.** for any prime ϵ-ideal P in $C^*(X)$ the family $CoE(P^c)$ is co-ϵ-filter on X. With $CoE(P^c) = \{CoE_\epsilon(f): f \in P^c, \epsilon > 0\}$.

Proof. Consider f belonging to P and $\epsilon > 0$, being P proper implies that it includes 0, so, 0 does not belong to P^c which means $\emptyset$ does not belong to $CoE(P^c)$. If $CoE_\epsilon(f)$ and $CoE_\varepsilon(g)$ belong to $CoE(P^c)$ it follows that f and g are not in P, by primeness fg will be in P^c, thus $CoE_\epsilon(f) \cap CoE_\varepsilon(g) \subseteq CoE_{\epsilon\varepsilon}(fg)$ is belong to $CoE(P^c)$. If $CoE_\epsilon(f) \subseteq CoE_\varepsilon(g)$, for some $f \notin P$ and $g \in C^*(X)$, since P is prime by Lemma 2.9, and lemma 2.10 absolutely convex, it follows that, $f^2 + g^2 \in P^c$, then $CoE_{min(\epsilon^2,\varepsilon^2)}(f^2 + g^2) \in CoE(P^c)$ and by proposition2.8, $CoE_\varepsilon(g) \subseteq CoE_\epsilon(f) \cup CoE_\varepsilon(g) \subseteq CoE_{min(\epsilon^2,\varepsilon^2)}(f^2 + g^2)$. Thus $CoE_\varepsilon(g) \in CoE(P^c)$.

For any $CoE_\epsilon(f) \in CoE(P^c)$, and any $\varepsilon > 0$, so $CoE_\varepsilon(f) \in CoE(P^c)$, and this follows because, $f \notin P$, as well as, $CoE_\epsilon(f)$ is cozero-set including $CoE_\varepsilon(f)$ when $\varepsilon \geq \epsilon$.

In the $C^*(X)$, as in the case of $C(X)$, the behavior of the Coz-filter contrasts with $Z(I)$ specifically, $Z(I)$ forms a z-filter as long as I is an ideal, primeness is not a necessary condition for this, the condition of primeness becomes relevant only when considering the relationship to a corresponding multiplicative set, a connection that will be clarified in the discussion that follows, the following example will explain this point.

2.13 Example. Consider the space of real numbers $\mathbb{R}$ with usual topology, and $C^*(\mathbb{R})$ be the ring on $\mathbb{R}$, define $I \subseteq C^*(\mathbb{R})$.

$$I = \{f \in C^*(\mathbb{R}) : f(x) = 0 \; for \; all \; x \geq 0\}$$

So, I is the set of bounded continuous functions that vanish on the right half-line $[0, \infty)$, clearly, I is an ideal of $C^*(\mathbb{R})$, but not prime, because the product of two nonzero functions (each vanishing on part of $[0, \infty)$) can lie in I without either factor being in I, $Z(I)$ is a z-filter, $Z(I) = \{Z(f): f \in I\}$, for example, take.

$$f(x) = \begin{cases} 0 & x \geq 0 \\ \sin(x) & x < 0 \end{cases} \quad \text{thus } f \in I$$

and $Z(f) = [0, \infty) \cup \{x < 0: sin(x) = 0\}$, so $Z(f)$ includes $[0, \infty)$ and hence all such zero sets contain $[0, \infty)$, or at least a neighborhood of it, clearly, $Z(I)$ is closed under finite intersections and supersets of such sets a z-filter.

For $CoE(I^c)$ as a coz-ϵ-filter, consider two functions $f(x) = \chi(-\infty, 0)(x)$, the characteristic function of $(-\infty, 0)$ is bounded but not in I, since $f \neq 0$ on $[0, \infty)$, and $g(x) = \chi(0,1)(x)$ also bounded and not in I, then both $f, g \in I^c$, thus $CoE_\epsilon(f) = \{x: |f(x)| > \epsilon\} = (-\infty, 0)$ and $CoE_\varepsilon(g) = (0, 1)$ and their intersection, $CoE_\epsilon(f) \cap CoE_\varepsilon(g) = \emptyset$. But note, now try, $h = f^2 + g^2$.

$$h(x) = f^2(x) + g^2(x) = \begin{cases} 0 & x \in (-\infty, 0) \cup (0,1) \\ 1 & other \; wise \end{cases}$$

So h is nonzero only on the union of $(-\infty, 0) \cup (0, 1)$ and vanishes elsewhere, so $h \notin I$ (since it does not vanish on $[0, \infty)$), but $CoE_{min(\epsilon^2,\varepsilon^2)}(h) = (-\infty, 0) \cup (0, 1)$, this is not a subset of the intersection $CoE_\epsilon(f) \cap CoE_\varepsilon(g) = \emptyset$, thus, $CoE_{\min(\epsilon^2,\varepsilon^2)}(h) \nsubseteq CoE_\epsilon(f) \cap CoE_\varepsilon(g)$, in fact we have $CoE_\epsilon(f) \cap CoE_\varepsilon(g) = \emptyset \notin CoE\_\epsilon \; (I^c)$ but a filter can't contain the empty set, this shows that $CoE(I^c)$ fails to be closed under finite intersections unless primality is assumed.

3. Prime z-ϵ-Ideal

The z-ideal theory gave a stronger structure for investigations of prime ideal-z-filters and prime ideal-coz-filter correspondence, here we aim to define the z-ideal definition according to ϵ-condition (ϵ-ideal and ϵ-filter) in $C^*(X)$ we call it z-ϵ-ideal (or z-e-ideal), and see if it still effect on the relation or not. Naturally the z-ideal definition extends to $C^*(X)$ with the primary distinctness, it only consists bounded functions, when in C may there are functions that are just continuous not bounded, the formal structure of the definition remains the same, but it indicates ideals that are fixed with zero-ϵ-sets of their functions.

3.1 **Definition.** the z-e-ideals follow same pattern as z-ideal but with e-sets, that is, an ideal in C^* is z-ϵ-ideal, whenever f, g be in $C^*(X)$ and $E_\epsilon(f) \in E(I)$ then there is g in I such that $E_\epsilon(f) \subseteq E_\varepsilon(g)$ imply $f \in I$.

In fact, $M^* \leftrightarrow E(M^*)$ is a bijective association among Maximal ideals M^* in C^* and e-ultrafilters which is the direct analogue of maximal ideal- z-ultrafilter connection in $C(X)$, as well between z-ultrafilter and e-ultrafilter via $U \mapsto E(E^-(U))$, noting that a z-ultrafilter U is related to the e-ultrafilter $E(E^{\leftharpoonup}\ (U))$ see [1] 2L, 13-15.

3.2 **Lemma.** The relation between e-ideals and z-ideals in C^* confirms that every e-ideal is z-ideal but the converse may not hold in general.

Proof. If we use the common definition of z-ideal in $C(X)$, $(Z(f) = Z(g),\ f\ \in I \Rightarrow g\ \in I)$, by lemma 2.11, we have absolute convexity, the property $E_\epsilon(\ f\ ) \subseteq\ E_\varepsilon(g), f \in I \Rightarrow g\ \in\ I$ for $f\ , g\ \in C^*(X)$ can be shown to hold. Conversely, if we continue using the common definition of z-ideal in $C^*(X)$, then this statement is also true, the definition of an ϵ-ideal $(I = E^{-1}\ [E(I)])$ imposes stricter conditions related to the vanishing behavior of functions near their zero sets (obtained by $E_\epsilon(f)$) than the general z-ideal property based solely on the zero sets themselves $(Z(f))$.

3.3 **Example.** A common counterexample involves ideals in $C^*(\mathbb{N})$, the ideal $J = \left\{ f \in C^*(\mathbb{N}): \lim_{n\to\infty} f(n) = 0 \right\}$, (sequences converging to zero) is a z-ideal in the sense that if $Z(f) = Z(g)$ and $f \in J$, then $g \in J$, however, (as is clear in 2L,3 in [1]) the function $j = (\frac{1}{n}) \in J$ but $j \notin E^{-1}[E(J)]$, so J is not an ϵ-ideal.

3.4 **Lemma.** An e-ideal $I \in C^*(X)$ is z-e-ideal iff $E^-\big(E(I)\big) = I$

Proof. $\Rightarrow$Consider I be z-ideal in C^*, to show $I =\ E^{-1}(E(I))$, if f be in I that is $E_\epsilon(f) \in E(I) = \mathcal{F}$ by definition of E^{-1}, $f \in E^{-1}(E(I))$, so, $I \subseteq E^{-1}(E(I))$. And if $g \in E^{-1}(E(I))$ implies $E_\epsilon(g) \in E(I)$ and I is z-ideal, then $g \in I$, so $E^{-1}(E(I)) \subseteq I$, thus $I = E^{-1}(E(I))$.

$\Leftarrow$Now suppose $I = E^{-1}(E(I))$ then consider $E_\epsilon(f)$ belongs to $Z(I)$, there is a g in I with $E_\varepsilon(g) \subseteq E_\epsilon(f) \in E(I)$, so, $f \in E^{-1}(E(I))$ thus $f \in I$ since $I = E^{-1}(E(I))$.

Recall a 1-1 relationship between z-e-ideals I in $C^*(X)$ and e-filters $\mathcal{F}$ on X can be found, via $\mathfrak{A}(I) = \{E_\epsilon(f) | f \in I\}$ and $\mathfrak{B}(\mathcal{F}) = \{f \in C^*(X) | E_\epsilon(f) \in \mathcal{F}\}$, those maps are inverses, $\mathfrak{B}(\mathfrak{A}(I)) = \{f \in C^*(X) | E_\epsilon(f) \in \mathfrak{A}(I)\} = \{f \in C^*(X) | \exists g \in I\ with\ E_\epsilon(f) \supseteq E_\varepsilon(g)\}$, since I is a z-e-ideal this recover I. And $\mathfrak{A}(\mathfrak{B}(\mathcal{F})) = \{E_\epsilon(f) | f \in \mathfrak{B}(\mathcal{F})\} = \mathcal{F}$.

Through z-e-ideals many extra things could be said about e-ideal-co-e-filter relation in $C^*(X)$, the Them 2.12. in [1] and Them 2.5. in [2] shows the impact of the z-ideal theory on relationships and to be held must both sides be prime, same pattern in e-ideal-co-e-filter can be found, the next theorem explains more about it.

3.5 **Lemma.** ε-ideals are z-ideals, [1] and Lemma 1.18, every prime z-e-ideal P in $C^*(X)$ is a z-ideal, if $f \in P$ and $E_\epsilon(\ f\ ) \subseteq\ E_\varepsilon(g)$ then $g \in P$.

3.6 **Lemma.** Consider f, g be in P, also $CoE_\epsilon(f)$ and $CoE_\varepsilon(g)$ not belonging to $\mathcal{F}$ then $CoE_{(\epsilon+\varepsilon)}(f+g) \subseteq CoE_\epsilon(f) \cup CoE_\varepsilon(g)$.

Proof. Let x belong to $CoE_{\epsilon+\varepsilon}(f+g)$ so that $|f(x)+g(x)| > \epsilon + \varepsilon$, and it is obviouse $|f(x)| + |g(x)| > \epsilon + \varepsilon$, so, their sum is greater than $\epsilon + \varepsilon$, so, at least must $|f(x)| > \epsilon$ or $|g(x)| > \varepsilon$ gives that $x \in CoE_\epsilon(f)$ or $x \in CoE_\varepsilon(g)$, thus $x \in CoE_\epsilon(f) \cup CoE_\varepsilon(g)$.

3.7 **Theorem.** Consider non-compact Tychonoff space X, and P be prime e-ideal in $C^*(X)$ then there is bijective correspondence among prime z-e-ideal in $C^*(X)$ and co-e-filters on X given by.

(a) If P is a prime z-ϵ-ideal in $C^*(X)$ then the collection $Coz(P^c) = \{CoE_\epsilon(f) | f \notin P, \epsilon > 0\}$ is prime Co-ϵ-filter on X.

(b) If $\mathcal{F}$ is prime Co-e-filter on X and $M = CoE^{\leftharpoonup}(\ \mathcal{F}) = \{f \in C^*: \epsilon > 0, CoE_\epsilon(f) \in \mathcal{F}\}$,then the set $C^*\backslash M = P = \{f \in C^*: \epsilon > 0, CoE_\epsilon(f) \notin \mathcal{F}\}$ is a prime z-e-ideal in $C^*(X)$.

These maps are mutually inverse and restrict to the bijection between prime z-e-ideals and prime co-e-filters.

Proof.

(a) Clearly every ideal contains 0, so for any $f \notin P$, $f \neq 0$, this satisfies $CoE_\epsilon(f) \neq \emptyset$, then $\emptyset \notin CoE(P^c)$ since P is prime z-ϵ-ideal in C^* as well its proper, so $f \in C^*\backslash P$, implies $CoE_\epsilon(f) \in CoE(P^c)$. Consider $CoE_\epsilon(f) \in CoE(P^c)$ so $f \notin P$ and let $CoE_\epsilon(f) \subseteq CoE_\varepsilon(g)$ for some $g \in C^*(X)$, we must show $g \notin P$ by duality $CoE_\epsilon(f) \subseteq CoE_\varepsilon(g)$ is equivalent to $E_\varepsilon(g) \subseteq E_\epsilon(f)$, suppose for contradiction $g \in P$. Apply the z-ideal property (Lemma 3.5) with $g \in P$ and $E_\varepsilon(g) \subseteq E_\epsilon(f)$ conclude $f \in P$, this contradicts $f \notin P$, thus $g \notin P$, so $CoE_\varepsilon(g) \in CoE(P^c)$. Let $CoE_\epsilon(f), CoE_\varepsilon(g) \in CoE(P^c)$, so $f, g \notin P$ since P is prime and $f, g \notin P$, we have $fg \notin P$ by Proposition2.8, $CoE_{(\epsilon\varepsilon)}(fg) \supseteq CoE_\epsilon(f) \cap CoE_\varepsilon(g)$, since $fg \notin P$, this intersection belongs to $CoE(P^c)$ thus $CoE(P^c)$ is co-ϵ-filter, let $A \cup B \in CoE(P^c)$, $\min(\epsilon^2, \varepsilon^2) = \delta^2$ and $h = f^2 + g$^2, where $A = CoE_\epsilon(f)$ and $B = CoE_\varepsilon(g)$ are co-e-sets, from proposition2.8, $A \cup B \subseteq CoE_{\delta^2}(h)$ and $CoE_{\delta^2}(h) \in CoE(P^c)$ implies $f^2 + g^2$ is not belong to P, suppose both $f \in P$ and $g \in P$ then $f^2 = f \cdot f \in P$ (P is an ideal) and $g^2 = g \cdot g \in P$, so $f^2 + g^2 \in P$ a contradiction, therefore $f \notin P$ or $g \notin P$, giving $A \in CoE(P^c)$ or $B \in CoE(P^c)$ so $CoE(P^c)$ is Prime.

Since every function $f \in C^*(X)$ is bounded, every $CoE_\epsilon(f)$ is a co-ϵ-set of a bounded function, exactly for any $\varepsilon \in (0, \|f\|_\infty)$, we have $CoE_\epsilon(f) = coz_\varepsilon((|f| - \varepsilon)^+)$ where $(|f| - \varepsilon)^+ \in C^*(X)$ so $CoE(P^c)$ consists entirely of co-e-sets.

(b) Let $\mathcal{F}$ be a prime co-ε-filter on X then $\emptyset = CoE_\epsilon(0) \notin \mathcal{F}$ (by filter's axiom) then $0 \notin M$ gives $0 \in P$. Let $f \in P$ and $g \in C^*(X)$, then $CoE_\epsilon(f) \cap CoE_\varepsilon(g) \subseteq CoE_{(\epsilon\varepsilon)}(fg) \subseteq CoE_\epsilon(f)$, if $CoE_{(\epsilon\varepsilon)}(fg) \in \mathcal{F}$, then $CoE_\epsilon(f) \in \mathcal{F}$, contradicting $f \in P$, so $CoE_{(\epsilon\varepsilon)}(fg) \notin \mathcal{F}$ then $fg \notin M$, thus $fg \in P$. Let $f, g \in P$, so $CoE_\epsilon(f), CoE_\varepsilon(g) \notin \mathcal{F}$, by lemma 3.6, $CoE_{(\epsilon+\varepsilon)}(f + g) \subseteq CoE_\epsilon(f) \cup coz(g)$, suppose $CoE_{(\epsilon+\varepsilon)}(f + g) \in \mathcal{F}$ implies $CoE_\epsilon(f) \cup CoE_\varepsilon(g) \in \mathcal{F}$ by primeness of $\mathcal{F}$, $CoE_\epsilon(f) \in \mathcal{F}$ or $CoE_\varepsilon(g) \in \mathcal{F}$ but this contradicts $f, g \in P$, so, $CoE_{(\epsilon+\varepsilon)}(f + g) \notin \mathcal{F}$ that is $f + g \notin M$, giving $f + g \in P$. let u be unit in $C^*(X)$, so that $CoE_\epsilon(u) = X$, since F is non-empty then for every A in $\mathcal{F}$ we have A is subset of X which means X is also $\mathcal{F}$, so, $CoE_\epsilon(u) = X \in \mathcal{F}$, implies u is not belong to P, then it is proper. Let $fg \in P$, then $fg \notin M$ gives $CoE_{(\epsilon\varepsilon)}(fg) \subseteq CoE_\epsilon(f) \notin \mathcal{F}$, but $\mathcal{F}$ is Co-e-filter, so, $CoE_\epsilon(f) \notin \mathcal{F}$, same pattern $CoE_\varepsilon(g) \notin \mathcal{F}$ implies $f, g \notin M$ therefore $f, g \in P$ so P is prime. For $f, g \in C^*$, let $E_\varepsilon(g) \in E(P)$ then there exist $f \in P$ such that $E_\epsilon(f) \subseteq E_\varepsilon(g)$ i.e., $CoE_\varepsilon(g) \subseteq CoE_\epsilon(f)$, then f is not belong to M, so, $CoE_\epsilon(f) \notin F$ gives that $CoE_\varepsilon(g) \notin \mathcal{F}$ means $g \notin M$ implies $g \in P$, thus P is z-e-ideal.

$CoE\big(CoE^-(\mathcal{F})\big) = \mathcal{F}$ by the definition $CoE\big(CoE^-(\mathcal{F})\big) = \{CoE_\epsilon(f) : f \notin P\} = \{CoE_\epsilon(f) : CoE_\epsilon(f) \in \mathcal{F}\}$, so, every element of $\mathcal{F}$ is of this form $CoE_\epsilon(f)$ for some $f \in C^*(X)$ and this equals $\mathcal{F}$.

$CoE^-\big(CoE(P^c)\big) = P$ we need show $f \in P \Leftrightarrow CoE_\epsilon(f) \notin CoE(P^c)$, $\Rightarrow$ Suppose f is in P for any g not in P and let $CoE_\varepsilon(g) = CoE_\epsilon(f)$ i.e. $E_\epsilon(f) = E_\varepsilon(g)$, but P is z-e-ideal, and $f \in P$ with $Z(f) \subseteq Z(g)$ implies $g \in P$ contradicting $g \notin P$ so no function outside P has the same cozero set as f, thus $CoE_\epsilon(f) \notin \{CoE_\varepsilon(g) : g \notin P\} = CoE(P^c)$.

$\Leftarrow$ suppose f be a member of P, so that, $CoE_\epsilon(f)$ is belong to $\{CoE_\varepsilon(g) : g \notin P\} = CoE(P^c)$, so, $CoE_\epsilon(f) \in CoE(P^c)$, thus the maps are mutually inverse.

4. Co-ϵ-ultrsfilter

The notation of Coz-ultrafilters on $C(X)$ is well studied in [2], its relationship with minimal prime ideals also proved, comparing with the existing known relation z-ultrafilter-maximal ideal in [1], same pattern we extend this correspondence to $C^*(X)$ in non-compact space X, inspiring by the information's on e-ultrafilter and maximal ideals in 2L in [1], moreover exploring some Co-e-ultrafilters property. First begin with mentioning some problems in E which given detailed exploration on the subject.

4.1 Theorem. ([1]2L)

1. If $\mathcal{E}$ is an e-ultrafilter then $E^{-}(\mathcal{E})$ is a maximal ideal.
2. If M^* is a maximal ideal in C^* then $E(M^*)$ is an e-ultrafilter.
3. The maximal ideals are e-ideals in the ring C^*(X).

4.2 Proposition. (cf. [11], Ch. 3, Ex. 1; [18], Thm. 2), $C(X) \setminus P$ is multiplicative set whenever P be prime ideal in $C(X)$, moreover $C(X) \setminus M$ is minimal prime ideal iff its complement be maximal multiplicative set of $C(X)$ on completely regular (Tychonoff) space X.

An analogous description holds for $C^*(X)$, M is maximal multiplicative set on $C^*(X)$ with $0 \notin M$ iff $C^*(X) \setminus M$ is a minimal prime ideal ([18], Thms. 1 and 35, pp. 1, 24), more generally, this equivalence is valid for $C(X)$, where (X) is any completely regular space.

4.3 Corollary. The corresponding statement for $C^*(X)$, every minimal prime ideal of $C^*(X)$ is e-ideal is equivalent to the classical result that every minimal prime ideal of $C(X)$ is z-ideal which is established in [9] and [8], and this distinction among these two formulas comes from boundedness condition inherent in $C^*(X)$, which necessitates the use of e-ideals in place of z-ideals.

4.4 Lemma. for $f \neq 0$ in $C^*(X)$, $CoE_\epsilon(f) \neq \emptyset$ as soon as $\epsilon \geq \| f \|_\infty$, so, when we write $CoE_\epsilon(f) \in \mathcal{F}$ for a filter $\mathcal{F}$, ϵ is implicitly understood to range over $(0, \| f \|_\infty)$ exactly as the proof of theorem 3.7(a) already we assumed when it confirms $f \neq 0 \Rightarrow CoE_\epsilon(f) \neq \emptyset$, and this is needed for filter axioms to be logic, then we'll use this silently below.

Dualizing 2L.12 in [1] a maximal e-filter is called an e-ultrafilter to the cozero side, same as in Definition 1.8 dualizes 2L's E_ϵ-filters, then for Co-e-ultrafilter we have the following definition.

4.5 Definition (Co-e-ultrafilter). A Co-e-filter $\mathcal{U}$ on X is a Co-e-ultrafilter if it is not properly contained in any other Co-e-filter on X.

Note, every Co-e-filter extends to one, follows by the same Zorn's-Lemma argument as 2L.12, in [1] a chain of Co-e-filters has its union as an upper bound, which is again a Co-e-filter.)

4.6 Lemma. If $\mathcal{U}$ be Co-e-filter then $\mathcal{U}$ is Co-e-ultrafilter, if for every $f \in C^*(X)$, and $CoE_\epsilon(f) \notin U$, there exist $g \in C^*(X)$ and $\delta > 0$ with $CoE_\delta(g) \in U$ s.t $CoE_\epsilon(f) \cap CoE_\delta(g) = \emptyset$.

Proof. Suppose no $CoE_\delta(f)$ meets every $B \in \mathcal{U}$, for every $\delta > 0$ (by the Lemma 4.4, this is the same statement for every valid δ, since $CoE_\epsilon(f) \notin \mathcal{U}$ forces $CoE_\delta(f) \notin \mathcal{U}$ for all δ, a set can only be in $\mathcal{U}$ for one δ range if it's in for all, by the Co-e-filter axiom $CoE\big(CoE^{-1}(\mathcal{U})\big) = \mathcal{U}$ itself) form $\mathcal{G} = \{A \in Coz(X) : A \supseteq B \cap CoE_\delta(f) \text{ for some } B \in \mathcal{U}, \delta > 0\}$, every finite intersection of generators $B \cap CoE_\delta(f)$ is nonempty by our contradiction hypothesis, so $\emptyset \notin \mathcal{G}$, $\mathcal{G}$ satisfy the finite intersection and supersets axioms by construction, and it inherits the ϵ-generation property from $\mathcal{U}$ and from f itself, so, $\mathcal{G}$ is a Co-e-filter. Since $CoE_\epsilon(f) \in \mathcal{G} \setminus \mathcal{U}$, we get $\mathcal{G} \supsetneq \mathcal{U}$, contradicting that $\mathcal{U}$ is maximal.

This is exactly cozero counterpart of 2L.14 in [1] which described maximal ideals of C^* by analogous "meets every member" condition on the E_ϵ side.

4.7 Proposition. Every Co-e-ultrafilter $\mathcal{U}$ on X is a prime Co-e-filter.

Proof. Suppose $CoE_{\epsilon}(f) \cup CoE_{\delta}(g) \in \mathcal{U}$ but $CoE_{\epsilon}(f) \notin \mathcal{U}$ and $CoE_{\delta}(g) \notin \mathcal{U}$ by Lemma 4.6, pick h_1, γ_1 with $CoE_{\gamma_1}(h_1) \in \mathcal{U}$, $CoE_{\gamma_1}(h_1) \cap CoE_{\epsilon}(f) = \emptyset$ and $h_2, \gamma_2$with $CoE_{\gamma_2}(h_2) \in \mathcal{U}$, $CoE_{\gamma_2}(h_2) \cap CoE_{\delta}(g) = \emptyset$. Let $B = CoE_{\gamma_1}(h_1) \cap CoE_{\gamma_2}(h_2) \in \mathcal{U}$ then $B \cap (CoE_{\epsilon}(f) \cup CoE_{\delta}(g)) = \emptyset$, so, since both B and $CoE_{\epsilon}(f) \cup CoE_{\delta}(g)$ lie in $\mathcal{U}$, their intersection is $\emptyset \in \mathcal{U}$ contradicting the filter axiom, so $\mathcal{U}$ is prime.

Definition 2.6 gives $E_{\epsilon}(f) = X \setminus CoE_{\epsilon}(f)$, so $E_{\epsilon}(f) = E_{\epsilon}(g) \iff CoE_{\epsilon}(f) = CoE_{\epsilon}(g)$ trivially. Hence the z-ϵ-ideal condition of Definition 12.1 and the z-e-ideal condition used in Theorem 3.7 are the *same* condition on an ideal, read through complementary set-families; by Lemma 3.4 this class coincides with the ϵ-ideals, so.

4.8 Corollary. Every minimal prime ideal of $C^*(X)$ is a z-e-ideal.

(Immediate from Corollary 4.3 + Lemma 3.2 + the observation above.) Note Corollary 4.3's hypothesis is not vacuous, [1] Theorem 14.7 and its corollary (p. 197) prove directly that every minimal prime ideal of $C(Y)$is a z-ideal for *any* Tychonoff Y.

Finally, in the following theorem we establish the main result of this section.

4.9 Theorem.

(a) Let $\mathcal{U}$ be Co-e-ultrafilter then, $P = C^*(X) \setminus M = \{f \in C^*(X) | \exists \epsilon > 0 \ s.t. CoE_{\epsilon}(f) \notin \mathcal{U}\}$ is minimal prime ideal of $C^*(X)$, where $M = CoE^{-1}(\mathcal{U}) = \{f \in C^*(X) | CoE_{\epsilon}(f) \in \mathcal{U}\}$.

(b) Let P be minimal prime ideal in $C^*(X)$ then $CoE(P^c)$ is Co-e-ultrafilter on X, recall, $CoE(P^c) = \{CoE_{\epsilon}(f) | f \notin P, f \in C^*(X)\}$.

Proof.

(a) Let $\mathcal{U}$ be a Co-e-ultrafilter, $M = CoE^{-1}(\mathcal{U})$, $P = C^*(X) \setminus M$ by Proposition 4.7, $\mathcal{U}$ is a prime Co-e-filter, applying Theorem 3.7(b), P is a prime z-e-ideal of $C^*(X)$ with $CoE(P^c) = \mathcal{U}$. Now for the minimality among prime z-e-ideals, suppose $Q \subsetneq P$ is another prime z-e-ideal in $C^*(X)$, then $Q^c \supsetneq P^c$, and since CoE is monotone on this family $CoE(Q^c) \supseteq CoE(P^c)$, so $CoE(Q^c) \supseteq \mathcal{U}$, Theorem 3.7(a) guarantees that $CoE(Q^c)$ is prime Co-e-filter, in particular Co-e-filter, and $\mathcal{U}$ is Co-e-ultrafilter which means it is maximal among *all* Co-e-filters, since $\mathcal{U} \subseteq CoE(Q^c)$ and both are Co-e-filters, then the maximality of $\mathcal{U}$ forces $CoE(Q^c) = \mathcal{U}$, but the maps of Theorem 3.7 are mutually inverse, so $Q = C^*(X) \setminus CoE^{-1}(\mathcal{U}) = P$ contradicting $Q \subsetneq P$. So $Q = P$ means no prime z-e-ideal is strictly contained in P. Suppose some prime ideal $Q \subsetneq P$ existed without the z-e-ideal property. By G–J's Theorem 0.16 (apply it with $I = (0)$, $S = C^*(X) \setminus Q$), there is a prime ideal $Q_{\min} \subseteq Q \subseteq P$ minimal with the property of being contained in Q, standard chain arguments make $Q_{\min}$ a genuine minimal prime of $C^*(X)$. By Corollary 4.8, $Q_{\min}$ is a z-e-ideal, hence prime z-e-ideal satisfying $Q_{\min} \subseteq P$. By the minimality above forces $Q_{\min} = P$, but then $P \subseteq Q \subsetneq P$ is absurd. So no prime ideal at all is strictly contained in P, P is a minimal prime ideal of $C^*(X)$.

(b) Let P be minimal prime ideal of $C^*(X)$, $M = C^*(X) \setminus P = P^c$ by Corollary 4.8, P is prime z-e-ideal, so by Theorem 3.7(a), $CoE(P^c)$is prime Co-e-filter, it remains to upgrade prime to ultra, by Proposition 4.2 (applied to $C^*(X)$), $M = P^c$ is maximal multiplicative subset of $C^*(X)$ not containing 0. For the sake of contradiction suppose, $\mathcal{G} \supsetneq CoE(P^c)$ is Co-e-filter properly containing it let $\mathrm{N} = CoE^{-1}(\mathcal{G})$ for $f, g \in N$, $CoE_{\epsilon}(f), CoE_{\delta}(g) \in \mathcal{G}$ for suitable ϵ, δ, so their intersection lies in $\mathcal{G}$ (by filter axiom), and by Proposition2.8, $CoE_{min(\epsilon^2,\delta^2)}(fg) = CoE_{\epsilon}(f) \cap CoE_{\delta}(g) \in \mathcal{G}$, thus $fg \in N$. Also $0 \notin N$, since $\emptyset \notin \mathcal{G}$. So N is multiplicative, $0 \notin N$, and $\mathrm{N} \supseteq M$ (since $\mathcal{G} \supseteq CoE(M)$). By maximality of M, $\mathrm{N} = M$. But since $\mathcal{G}$ is a Co-e-filter, it satisfies its own defining closure axiom $CoE\big(CoE^{-1}(\mathcal{G})\big) = \mathcal{G}$, that's literally what "Co-e-filter" means. So $\mathcal{G} =$

$CoE(N) = CoE(M) = CoE(P^c)$, contradicting $\mathcal{G} \supsetneq CoE(P^c)$. So no Co-$e$-filter properly contains $CoE(P^c)$: $CoE(P^c)$is a Co-e-ultrafilter.

5. Conclusion

In this paper we extend the ideal–filter framework for $C^*(X)$ by submitting the notion of Co-ϵ-filters as the natural cozero-set counterpart of ϵ-filters on non-compact Tychonoff spaces, the developed theory establish that the boundedness condition inherent in $C^*(X)$ does not prevent the existence of a rich ideal–filter correspondence, rather, it requires an appropriate refinement through the ϵ-setting. Specially the bijections betwixt prime (z-e-ideals & Co-e-filters) and (minimal prime ideals & Co-ϵ-ultrafilters) that established here provide bounded analogues of the known relation on $C(X)$, the results unify the cozero-set and ideal-theoretic idea include bounded function ring, offering an area that parallels the well-known theory while mirroring the characteristic frame of $C^*(X)$, consequently, the present approach provides a basis for more investigations into algebraic and topological properties of bounded rings of continuous functions and their associated filter frames.

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