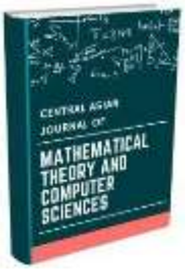

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Article

Gamma-Type Hesitant Fuzzy Ring

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Abstract: The main purpose of this research is to formally define a "γ -Hesitant Fuzzy Ring" and discuss the fundamental algebraic properties of it, including ideals and quotient rings, and to investigate the effect of the parameter γ on the degree of hesitancy. Traditional fuzzy algebraic models with an effective representation of states of "hesitancy" that result from several opinions or heterogeneous sources are lacking, since they assume that the membership value of each element is a single value. This constraint is a challenge for modelling in real-world applications where uncertainty is more complex. The main research question is What algebraic ring structure will allow for several "hesitancy values" to be assigned to elements of the ring, and the degree of this hesitancy to be controlled by a mathematical parameter, and the consistency of the algebraic operations (addition and multiplication) within the ring? The results indicate that the use of γ as an upper limit for the number of hesitant membership values results in a more flexible and adaptable algebraic structure. The authors also find that it is not easy to build a consistent membership function (closure under addition and multiplication) and it demands exact conditions to relate the membership values of different elements. The presented theorem shows strong connections between the membership values of an element and its additive inverse, which opens new possibilities for the development of the whole algebra of such structures and their applications in multi-criteria decision support systems.

Keywords: Hesitant Fuzzy Sets, Gamma-Type Hesitant Fuzzy Ring, Fuzzy Algebra, Decision Support Systems, Parameterized Fuzziness.

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1. Introduction

The theory of fuzzy sets, introduced by the Azerbaijani mathematician Lotfi Zadeh in 1965, represents a significant shift in how mathematics deals with uncertainty [1]. Instead of Aristotelian binary logic, it allowed for the representation of an element's degree of belonging as a value on the interval [0,1] [1]. This revolutionary concept has since expanded into numerous applications [2]. The following decades witnessed a substantial expansion in models of uncertainty. Atanassov introduced intuitionistic fuzzy sets, which incorporate the degree of non-belonging [3]. Rosenfeld established the concept of fuzzy groups for algebraic structures, and Mordeson and Malik followed suit with their work on commutative fuzzy algebra [4], [5]. Then Torra and Narukawa and Torra developed the hesitant fuzzy groups [6], [7].

Hesitant Fuzzy Sets) allow for multiple belonging values for each element, which aligns with the multiplicity of opinions in decision-making processes [8], [9].

This research attempts to synthesize algebraic concepts with this advanced type of fuzzyness by studying the "hesitant fuzzy loop of the gamma pattern." The novelty of this approach lies in introducing the gamma parameter (γ) as a tool to control the permissible "frequency" of elements, enabling the construction of more flexible mathematical models adaptable to different contexts of uncertainty and ambiguity [7], [8].

This coefficient represents an upper limit on the number of possible membership values for each element. If γ=1, we revert to the traditional fuzzy state [1]. If γ>1, we allow for higher degrees of frequency, reflecting multiple opinions and estimations [6], [9]. This research comprises six interconnected topics, beginning with the fundamentals of fuzzy and oscillatory set theory. It then defines the oscillatory loop of the Kama type and examines its basic structural properties. Next, it delves into the concept of idealisms and division algebras, and finally, it explores some potential applications [4], [5]. From Traditional to Fuzzy Oscillatory Set Theory

1- Traditional Sets to Hesitant Fuzzy Sets

2- Traditional Sets (Crisp Sets)

Traditional sets are the foundation upon which modern mathematics is built. In this classical framework, the relationship between an element and a set is a deterministic binary relationship: either the element belongs to the set or it does not belong [1]. Mathematically, the membership function (Characteristic Function) of set A can be defined as follows [4].

Definition 1:2: If A is a subset of a universal set A, then the membership function $tA: X \to [0,1]$ is defined as follows: [2]

$tA(x) = 1$ if was$x \in A$

$tA(X) = 0$ if was $x \notin A$

Example 2.2: Let $X = \{2,3,4,5,6,7,8,9,10\}$ be the set of numbers from 2 to 10 . Let A be the set of even numbers in X. Therefore: $A = \{2,4,6,8,10\}$ and the membership function is: [1].

$$tA(2) = 1, t(3) = 0, tA(4) = 1, tA(5) = 0, tA(6) = 1,$$
$$tA(7) = 0, tA(8) = 1, tA(9) = 0, tA(10) = 1$$

This binary logic works perfectly when the boundaries between categories are clear and decisive. However, the problem arises when we encounter vague or precisely undefined concepts [10].

3 - Fuzzy Sets

To overcome the limitations of binary logic, Lotfi Zadeh introduced the fuzzy sets theory in 1965. The core idea is to extend the domain of the membership function from the set $\{0,1\}$ to the entire closed interval $[0,1]$ [1].

Definition 3.1: A fuzzy set F in a universal set X is defined by a membership function $\mu F: X \to [0,1]$ where $\mu F(x)$ represents the membership function from X to F. Values close to 1 mean high membership, values close to 0 mean low membership, and intermediate values express moderate degrees of membership. [1], [2].

Example 3.2: Let's return to the hot weather example. A "hot" fuzzy set can be defined with a membership function as follows:

$\mu F(x) = 0$ IF was $x \leq 15$

$\mu F(x) = \frac{x-15}{15} > 15$ If was $x < 30$

$\mu F(x) = 1$ if was $x \geq 30$

Now, a temperature of 25°C gives a membership value

$\mu F(25) = 25 - 15/15 = 10/15 \approx 0.67$, which means that the weather is "somewhat hot" or "hot with a degree of 0.67". This representation is more accurate and reflects the gradual nature of the concept [10].

Example 3.3 (Large Numbers): Let $X = \{1,2,3 \ldots\ldots,10\}$. We define a "large" fuzzy set with the membership function:

$\mu Large(1) = 0$, $\mu Large(2) = 0.1$, $\mu Large(3) = 0.2$

$\mu Large(4) = 0.3$, $\mu Large(5) = 0.5$, $\mu Large(6) = 0.6$

$\mu Large(7) = 0.7,$ $\mu Large(8) = 0.8$, $\mu Large(9) = 0.9$, $\mu Large(10) = 1$

This function reflects that the number 10 is completely large, while 5 is moderately large with a degree of (0.5) [1].

4- Operations on Fuzzy Sets

Classical operations (union, intersection, complement) extend to fuzzy sets in natural ways:

Definition 4.1: Let F, G be two fuzzy sets in X with membership functions $\mu F, \mu G$. We define:

$$\mu F \cup G(x) = \max(\mu F(x), \mu G(x))$$

$\mu F \cap G(x) = \min(\mu F(x), \mu G(x))$

$\cdot \mu F^c(x) = 1 - \mu F(x)$

Example4:2 From the previous example ,if we define a small fuzzy set as follows:

$\mu Small(1) = 1, \mu Small(2) = 0.9, \mu Small(3) = 0.8, \mu Small(4) = 0.7,$
$\mu Small(5) = 0.5, \mu Small(6) = 0.4$, $\mu Small(7) = 0.3,$

$\mu Small(8) = 0.2, \mu Small(9) = 0.1$, $\mu Small(10) = 0,$

The intersection of "large" and "small" gives: (Dubois & Prade, 1980)

$\mu Large(5) \cap \mu Small(5) = \min(0.5,0.5) = 0.5$

$\mu Large(7) \cap Small(7) = \min(0.7,0.3) = 0.3$.

5- Hesitant Fuzzy Sets

Despite the power of fuzzy set theory, it assumes that the process of assigning a degree of belonging is a single, definite operation. But what if we have several experts, each with a different opinion on the degree of belonging of a particular element? Here, "hesitation" emerges as a natural consequence of multiple opinions [7].

Definition 5:1 The hesitant fuzzy set H in the global set X is defined by a function

$H: X \to P[0,1]$ ‹if $p[0,1]$

The set of all non-empty subsets of the interval [0,1]where the value $H(x)$ is called the "frequent belongings set" of the element x.

In other words, instead of returning a single value, the function returns a range of possible values [6], [8].

Example 5:2 (Employee Performance Evaluation): A panel of three experts evaluates an employee's performance on a scale of 0 to 1. The experts gave the following ratings:

Expert 1 = 0.8, Expert 2 = 0.7, Expert 3 = 0.9

In the fuzzy ensemble model, we express this situation as follows

$$H(employee = \{0.7,0.8,0.9\})$$

This representation retains all opinions without forcing a single choice or calculating an average that might lose important information.

Example 5:3: Let Xbe the set $X = [x_1, x_2, x_3]$. The membership function of the frequencies $\emptyset$ is as follows:

$\emptyset(x_1) = [0.5,0.7,0.9]$, $\emptyset(x_2) = [0.2,0.5,0.6]$, $\emptyset(x_3) = [0.4,0.7,0.8]$

The hesitant fuzzy set can be written as follows:

$\boldsymbol{H = \{< x_1, \{0.5 ، 0.7 ، 0.9\} >< x_2, \{0.2 ، 0.5, 0.6 > < x_3, \{0.4 ، 0.7 ، 0.8 >}$

6 - Theoretical Basis - Formal Definition of a Gamma-type Hesitant Fuzzy Ring (γHesitant Fuzzy Ring) with Examples

Reminder of the Classical Ring

Before defining the oscillating structure, it is necessary to recall the definition of the classical ring in abstract algebra.

Definition 6:1 The ring $(R, +,\cdot)$is a set R equipped with two binary operations: addition $(+)$ and multiplication $(\cdot)$. These operations satisfy:

1. $(R,\cdot)$ is an Abelian (commutative) group:

• Associative: $a + (b + c += (a + b) + c$

• Commutativity: $a + b = b + a$

• Existence of an identity element: There exists $0 \in R$ such that

$$a + 0 = a$$

• Existence of an additive inverse: For every $a \in R$ there exists -a ∈R such that

$$-a + a = 0$$

2. $(R, +)$Half-group

• Associativeness:$a \cdot (b \cdot c) = (a \cdot b) \cdot c$

3. Distributive property:

$a \cdot (b + c) = a \cdot b + a \cdot c$

$(a + b) \cdot c = a \cdot c + b \cdot c$

Example 6:2 $(Z, +,\cdot)$Integers with ordinary addition and multiplication form a ring [11].

7- The concept of the "gamma pattern"

The coefficient γ (gamma) is a positive natural number that represents the maximum number of allowed membership values for each element in the oscillating set. That is, for each element x, the number of values in $H(x)$ must be $\gamma \geq 1$, and the significance of γ is: [8].

If $\gamma = 1$, the structure reverts to a traditional fuzzy loop (each element has a unique affiliation value).

If $\gamma > 1$, higher frequencies are allowed.

γ can be chosen based on the nature of the problem (e.g., $\gamma =$ the number of experts, or $\gamma = 3$ to represent lower, middle, and upper values).

Definition7:1 The oscillatory fuzzy set of type γ on the set X is a function

$H: X \to P[0,1]$ where $P[0,1]$ is the set of all non-empty subsets of [0,1], and satisfies the condition: $H(x) \leq \gamma, \forall x \in X$. [6].

Example 7:2 If $\gamma = 3$, then the set {0.2,0.5,0.8} is acceptable, as are {0.3,0.7} and {0.4}. However, the set {0.1,0.3,0.5,0.9} is not acceptable because the number of its elements is $3 < 4$ [7]

8- Definition of a Gamma-Mode Hesitant Fuzzy Ring

Definition 8:1 Let $(R,+,\cdot)$ be a classical ring. We define a "gamma-mode hesitant fuzzy ring" (γ) as an ordered pair $(R,\emptyset)$ where $\emptyset: R \rightarrow P[0,1]$ is a γ −mode belonging function that satisfies the following conditions for all $x, y \in R$: [5], [4]

Condition 1 (Gamma-Mode): $\emptyset(x) \leq \gamma$ (the number of belonging values for each element does not exceed γ).

Condition 2 (Modeful Addition): $\emptyset(x+y)$ must be compatible with $\emptyset(x)$ and $\emptyset(y)$. We say that $\emptyset(x+y)$ is the "closed set" of the set:

$S - add = \{\min(\alpha+\beta, 1) \ni \alpha \in \emptyset(x), \beta \in \emptyset(y)\}$

A shrinking function can be applied if $S - add > \gamma$ to obtain a set with no more than γ elements. The shrinking function can be set by choosing a maximum, minimum, or average value for γ.

Condition 3 (Repeated Multiplication): $\emptyset(x \cdot y)$ must be compatible with $\emptyset(x)$ and $\emptyset(y)$. We say that $\emptyset(x \cdot y)$ is the "closed set" of the set:

$$S - mul = \{\alpha \cdot \beta \ni \alpha \in \emptyset(x), \beta \in \emptyset(y)\}$$

A shrinking function can be applied if $S - mul > \gamma$.

Condition 4 (Identity Elements): $\emptyset(R_0) and\ \emptyset(R_1)$, where $R_0 and\ R_1$ are the additive and multiplicative identity elements of R, must satisfy special conditions. We usually require: [11], [12]. $\emptyset(R_1) = \{1\}(1)$ (or a set containing)

$\emptyset(R_1) = \{1\}$ (1or a set containing)

9- Detailed Examples of Recurring Loops

Example 9.1 (Simplest example $\gamma = 1$):

Let $R = Z_2 = \{0,1\}$ with addition and multiplication pattern 2. We define $\emptyset$ as follows:

$\emptyset(0) = 1$

$\emptyset(1) = \{0.5\}$

This represents a recurring loop of pattern $\gamma = 1$ (because $\emptyset(x) = 1$ for all x). It is actually a classic fuzzy loop [1], [4].

Example 9:2: ($\gamma = 2$ over Z_3)

Let $R = Z_3 = \{0,1,2\}$ with operations of pattern 3. We know $\emptyset$:

$\emptyset(0) = \{1\}$

$\emptyset(1) = \{0.3,0.7\}$

$\emptyset(2) = \{0.5,0.8\}$

We need to check the addition and multiplication conditions.

Check that $0 = 2 + 1$ (in Z_3):

$S - add = \{\min(\alpha+\beta, 1) \ni \ \alpha \in \{0.3,0.7\}, \beta \in \{0.4,0.6\}\}$

$= \min(0.3 + 0.4,1)\,, \min(0.3 + 0.6,1)\,, \min(0.7 + 0.4,1),$

$\min(0.7 + 0.6,1) = \{1,1,0.9,0.7\} = \{1,0.9,0.7\}$

This set has 3 elements where $\gamma = 2$. We need to reduce it. Let's use the "choice of extremes" reduction: we choose the two largest values, $\{1,0.9\}$. However, $\emptyset(0)$ should be $\{1\}$ by definition. This is a contradiction! Therefore, this definition of $\emptyset$ is incorrect. We will try another definition [7], [5].

Example 9:3: (Correct definition of $\gamma = 2$ on Z_3):

$\emptyset(0) = \{1\}$ $\emptyset(1) = \{0.2,0.5\}$ $\emptyset(2) = \{0.5,0.8\}$

We verify that $1 + 2 = 0$

$$S - add = \{\min(0.2 + 0.5,1), \min(0.2 + 0.8,1), \min(0.5 + 0.5,1), \min(0.5 + 0.8,1\}$$
$$= \{1,1,1,1\}$$

This contains only two elements, and a reduction can be applied to make it $\{1\}$ (by selecting the maximum value) to match $\emptyset(0)$. This is better.

We verify the addition of $1 + 1 = 2$:

$$S - add\{(\min(0.2 + 0.2,1), \min(0.2 + 0.5,1), \min(0.5 + 0.2,1), \min(0.5 + 0.5,1)\}$$
$$= \{0.4,0.7,0.7,1\} = \{0.4,0.7,1\}$$

Reducing to two values: $\{0.7,1\}$ (which contradicts $\emptyset(2) = \{0.5,0.8\}$. This means that this determination is also incorrect. We see that constructing a consistent $\emptyset$ requires care [8].

Example 9:4: (Successful example - trivial loop):

Let R be a loop with only one element (the zero loop). Any $\emptyset$ can be defined that satisfies $\emptyset(0) \leq \gamma$because all operations result in 0. This is a trivial but valid example [4].

Example 9:5: (Example of a fuzzy, recurring Boolean ring):

Let $R = [0,1]$ be a Boolean ring where$\min(a, b) = a \wedge b\ and\ \max(a, b) = a \vee b$. We know $\emptyset$:$\emptyset(1) = \{0.1,0.5,0.9\} with\ \gamma = 1$

$1 \wedge 1 = 1$ Verified:

$$S - mull = \{\alpha \cdot \beta \backslash \alpha, \beta \in \{0.1,0.5,0.9\}\}$$

$$= \{0.01,0.05,0.09,0.05,0.25,0.45,0.09,0.45,0.81\}$$

$$= \{0.01,0.05,0.09,0.25,0.45,0.81\}$$

Reducing to 3 values (the three largest): we choose $\{0.25,0.45,0.81\}$. These are not equal to $\emptyset(1) = \{0.1,0.5,0.9\}$

Therefore, this selection is incorrect. We need to choose $\emptyset(1)$ so that it is closed under multiplication. [5].

Example 9:6: (A simple and successful example over Z_2):

Let $R = Z_2 = \{0,1\}$. Defien $\emptyset \ni \emptyset(0) = \{1\}$

$\emptyset(1) = \{0.2,0.8\}$ We verify that $1 + 1 = 0$ (in Z_2)

$$S - add = \{\min(0.2 + 0.2,1), \min(0.2 + 0.8,1), \min(0.8 + 0.8,1)\} = \{1,0.4\} = \{1,1,1,0.4\}$$

Applying a reduction to the largest value: $\{1\}$, which matches $\emptyset(0)$, we verify that $1 \cdot 1 = 1$

$$S - mul = \{0.2 \cdot 0.2, 0.2 \cdot 0.8, 0.8 \cdot 0.2, 0.8 \cdot 0.8\}\{0.04,0.16,0.16,0.64\} = \{0.16,0.64\}$$

0.16,0.64}: Reducing to two (larger) values

Therefore, this example is also incorrect.$\emptyset(1) = \{0.2, 0.8\}$

We see that constructing a consistent $\emptyset$ is not easy. This shows the importance of careful theoretical study.

Additive Identity Property

Theorem: Let $(R, \emptyset)$by $\gamma - ring$ if $\emptyset(0) = 1$, then for every $x \in R$, the set $\emptyset(x)$ must be closed under some transformation associated with the additive inverse [4], [5].

Proof:

We have $x + (-x) = 0$. From the reciprocal addition condition, $\emptyset(0)$ must be compatible with $\emptyset(x)$ and $\emptyset(-x)$. Applying the reciprocal addition, we obtain the set $S = \{\min(\alpha + \beta, 1), \alpha \in \emptyset(x), \beta \in \emptyset(-x)\}$. $\emptyset(0) = 1$ must be a subset of S after reduction (or equal to it after reduction). This implies that for every $\alpha \in \emptyset(x)$, there exists $\beta \in \emptyset(-x)$ such that $\min(\alpha + \beta, 1) = 1$, i.e., $\alpha + \beta \geq 1$. Similarly, for every

$\beta \in \emptyset - (x)$, there exists $\alpha \in \emptyset(x)$ such that $\alpha + \beta \geq 1$.

[7], [8] This relation connects the $\emptyset(x)$with $\emptyset(-x)$.

Example: In our example 1:9 $\emptyset(1) = \{0,1\}$. We are looking for $\emptyset(-1)$. In Z_2, $-1 = 1$ because $1 + 1 = 0$. Therefore,$\emptyset(-1) = \emptyset(1) = \{0,1\}$. We check the condition: For every $\alpha \in [0,1]$, does there exist $\beta \in [0,1]$such that $\alpha + \beta \geq 1$

•If $\alpha = 0$, we take $\beta = 1$, which gives$1 + 0 = 1 \geq 1$.

•If $\alpha = 1$, we take $\beta = 0$, which gives $1 + 0 = 1 \geq 1$. Therefore, the condition is satisfied [11].

2. Methodology

This research employs an axiomatic algebraic construction approach to formally define and analyze the structural properties of the γ-Hesitant Fuzzy Ring. The research methodology is structured systematically as follows:

1. **Axiomatization and Structural Formulation:**

Formulate the fundamental algebraic conditions for a γ-hesitant fuzzy ring over a classical ring $(R, +, \cdot)$, enforcing cardinality constraints such that $|\phi(x)| \leq \gamma$. [13]

2. **Definition of Closure and Reduction Operators:**

Define element-wise algebraic sum (S_{add}) and product (S_{mul}) sets [14]. To handle cases where $|S| > \gamma$, a reduction mapping Red_γ based on upper-bound/extreme value selection is established to maintain γ-cardinality bounds.

3. **Algebraic Consistency Analysis on Finite Rings:**

Evaluate hesitant membership functions on finite ring structures (such as $\mathbb{Z}_2$ and $\mathbb{Z}_3$) to assess closure properties under addition and multiplication operations.

4. **Derivation of Algebraic Theorems:**

Deduce formal mathematical proofs regarding additive identity elements, additive inverses, hesitant fuzzy ideals, and zero-divisor properties [15].

3. Results

A. Formal Definition of γ-Hesitant Fuzzy Ring

Let $(R, +, \cdot)$ be a classical ring. An ordered pair (R, ϕ) is called a **γ-Hesitant Fuzzy Ring** if the hesitant membership function $\phi: R \to \mathcal{P}([0,1]) \setminus \emptyset$ satisfies the following conditions for all $x, y \in R$:

1. **Gamma Cardinality Condition:** $|\phi(x)| \leq \gamma$, where $\gamma \geq 1$ is a fixed positive integer parameter.
2. **Additive Compatibility:** $\phi(x + y) = \mathrm{Red}_\gamma(S_{\mathrm{add}})$, where:
$$S_{\mathrm{add}} = \{\min(\alpha + \beta, 1) \mid \alpha \in \phi(x), \beta \in \phi(y)\}$$
3. **Multiplicative Compatibility:** $\phi(x \cdot y) = \mathrm{Red}_\gamma(S_{\mathrm{mul}})$, where:
$$S_{\mathrm{mul}} = \{\alpha \cdot \beta \mid \alpha \in \phi(x), \beta \in \phi(y)\}$$
4. **Identity Element Constraints:** $\phi(0) = \{1\}$ and $1 \in \phi(1_R)$.

B. Consistency Analysis on Finite Rings

- **Example 1 (Trivial Case $\gamma = 1$ on $\mathbb{Z}_2$):**

Let $R = \mathbb{Z}_2 = \{0,1\}$. Define $\phi(0) = \{1\}$ and $\phi(1) = \{0.5\}$. For $\gamma = 1$, all closure conditions are directly satisfied, and the structure collapses to a standard Zadeh fuzzy ring.

- **Example 2 (Constructive Limitations on $\mathbb{Z}_3$ for $\gamma = 2$):**

Let $R = \mathbb{Z}_3 = \{0,1,2\}$. Define $\phi(0) = \{1\}$, $\phi(1) = \{0.3, 0.7\}$, and $\phi(2) = \{0.5, 0.8\}$.

Evaluating $0 = 2 + 1 \pmod 3$ yields:

$$S_{\mathrm{add}} = \{\min(\alpha + \beta, 1) \mid \alpha \in \{0.3, 0.7\}, \beta \in \{0.5, 0.8\}\} = \{0.8, 1.0, 0.9, 1.0\} = \{0.8, 0.9, 1.0\}$$

Since $|S_{\mathrm{add}}| = 3 > \gamma$, applying the reduction function $\mathrm{Red}_2(S_{\mathrm{add}})$ yields $\{0.9, 1.0\}$. However, this contradicts the initial definition $\phi(0) = \{1\}$. This result proves that hesitant membership values cannot be assigned arbitrarily; they must strictly satisfy the closure equations of the underlying ring.

C. Additive Inverse Theorem

Theorem 1 (Additive Inverse Relationship):

Let (R, ϕ) be a γ-hesitant fuzzy ring with $\phi(0) = \{1\}$. For every $x \in R$ and each $\alpha \in \phi(x)$, there exists $\beta \in \phi(-x)$ such that $\alpha + \beta \geq 1$.

Proof:

Since $x + (-x) = 0$, additive compatibility requires that $\phi(0) = \{1\} \subseteq \mathrm{Red}_\gamma(S_{\mathrm{add}})$. Thus, at least one pair $(\alpha, \beta) \in \phi(x) \times \phi(-x)$ must satisfy $\min(\alpha + \beta, 1) = 1$, which implies $\alpha + \beta \geq 1$. ■

4. Discussion

The findings reveal several significant theoretical and practical insights:

1. **Algebraic Rigidity vs. Flexibility of Parameter γ:**

Increasing γ provides greater flexibility for capturing multi-expert hesitation. However, as demonstrated in Example 2, the algebraic structure imposes strict constraints on valid membership combinations. This underscores the need for algorithmic methods to generate consistent membership functions automatically.

2. **Generalization of Fuzzy Structures:**

The γ-hesitant fuzzy ring successfully generalizes classical fuzzy rings (when $\gamma = 1$). Introducing the reduction operator Red_γ serves as a critical bridge between unconstrained multi-valued hesitant sets and real-world decision bounds.

3. **Application to Multi-Criteria Group Decision-Making (MCGDM):**

In group decision support systems, γ can be tailored to the number of expert bodies or belief bounds (e.g., lower, middle, and upper bounds), preserving original evaluation data without relying on lossy averaging techniques.

5. Conclusion

This study formally introduced and evaluated the **γ-Hesitant Fuzzy Ring** structure. By parameterizing hesitant membership through γ, this model bridges classical fuzzy rings and complex multi-valued hesitant algebraic systems.

Key conclusions include:

- Additive and multiplicative compatibility requires a well-defined reduction operator when operational cardinality exceeds γ.
- Constructing valid membership functions ϕ is strictly constrained by the addition and multiplication tables of the underlying ring.
- A necessary mathematical relationship between an element's membership values and those of its additive inverse was proven ($\alpha + \beta \geq 1$).

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