Journal for Technology and Science
Homepage : https://journal.antispublisher.com/index.php/IPTEKS/index
Email : admin@antispublisher.com

e-ISSN : 3047-4337
JOTECH, Vol. 3, No. 3, July 2026
Page 28-36

Spectral Decomposition Strategies for Anomalous Diffusion and Impulsive Boundary Value Systems

Nada Abdul-Hassan Atiyah
Department of Mathematics, College of Education, Al-Qadisiyah University, Iraq

DOI : https://doi.org/10.61796/ipteks.v3i3.512

Sections Info
Article history:
Submitted: March 23, 2026
Final Revised: April 11, 2026
Accepted: May 16, 2026
Published: June 25, 2026
Keywords:
Spectral decomposition
Anomalous transport
Non-integer operators
Impulsive dynamics
Frequency domain analysis
Oscillatory artifacts
Fundamental solutions

ABSTRACT
Objective: *Physical models with memory or impulsive effects of non-local nature present complications mathematically.* ***Method:*** *To circumvent such computational challenges, a frequency domain approach is explored herein. The new algorithm allows one to deal easily with fractional derivatives and strong singularities by reducing the problem of solving the integro-differential equation to the solution of an algebraic problem.* ***Results:*** *Numerical analysis suggests that the transition to the frequency domain is indeed valid but exhibits a noticeable reduction in rate of convergence, which goes from $O(N^{-2})$ in normal situations to $O(N^{-\alpha})$ for the anomalous case. We further consider sustained oscillations induced by impulse-like forcing.* ***Novelty:*** *The traditional approach using finite difference method is known to have stability issues when dealing with anomalous diffusion as well as impulses like Dirac delta functions.*

INTRODUCTION

The behavior of the systems with some constraints has always been an important aspect of applied mathematics researches. The investigation of the thermal dispersion process, fluid dynamics, and structural stability requires the use of differential equations subject to some constraints [1, 2]. The solution of the governing equation according to the given constraints should be found [3, 4]. Mathematicians have been employing integer calculus for the solving of these problems during many years already. However, many engineering problems cannot be solved using traditional approaches nowadays.

Consider materials with memory, which means that they can remember previous deformations or fluids, which demonstrate non-Newtonian flow. These aspects cannot be considered by means of ordinary derivatives. Consequently, it is necessary to employ fractional calculus. This branch of mathematics has become something more than pure scientific interest nowadays because it is indispensable [5, 6]. Unlike derivatives, integration operators calculate the entire process of transformation, not just one moment of time [7, 8]. That is why they are so valuable in the modeling of the behavior of viscoelastic materials and in signal processing [9, 10, 11]. Unfortunately, introducing these features into the system with some constraints complicates the problem significantly.

Moreover, the problem becomes even more challenging if the system is governed by stochastic or impulse control. For instance, in practice, energy input occurs through impulses, whose mathematical representation requires the usage of certain distributions, including the famous Dirac delta function [12]. Because impulses control the reaction, a traditional continuous solution concept is inappropriate, and one should search for the solution among distributions [13, 14]. Even though this process has an established foundation within functional analysis, its numerical realization remains a hard nut to crack.

The frequency domain transforms offer an additional powerful tool to avoid the difficulties of the spatial domain. One need not tackle a differential operator at all but project the function under consideration along with the excitation source on a set of orthogonal harmonics [16, 17]. Thus, complex integro-differential equations can be simplified into algebraic problems [18]. Additionally, harmonic decomposition automatically introduces generalized functions. In light of that, because the spectral representation of the delta function is known, point loads can be modeled without suffering from computational instability related to grids [19].

The applicability of the frequency domain method hinges on the decay speed of truncation error. For ordinary systems whose input functions are smooth, the decays happen quickly to result in a superb bound of $O(N^{-2})$. In contrast, non-integer order derivatives impose limits on smoothness of solutions to slow down the decays to $O(N^{-\alpha})$. To characterize such an obstacle formally, we present the following proposition.

Proposition 1 (Truncation Decay)

Let w be the exact solution of the anomalous equation $D^\alpha w = h$ on (0,1) satisfying homogeneous boundary conditions. Assume that the source term h is sufficiently smooth. Then the L^2 error of its spectral truncation w_N can be bounded by $||w - w_N||_{L^2} \le K N^{-\alpha}$, where K is a positive constant.

Proof Sketch

For the exact solution w of the problem, the spectral weight satisfies $w_n = h_n / \mu_n^\alpha$ with $\mu_n = (n\pi)^2$. Using the law of energy conservation, we obtain the bound of $\sum_{n=N+1}^\infty (h_n/(n\pi)^{2\alpha})^2$ for the squared error. Bounding the coefficients of the source term and switching to an integral lead to an asymptotic behavior of $O(N^{1-2\alpha})$. Taking the square root results in the convergence rate $O(N^{-\alpha+1/2})$.

RESEARCH METHOD

Structural Architecture and Methods

System Constraints and Boundaries

We begin by establishing the physical domain. Let $\Gamma \subset \mathbb{R}^n$ represent a finite region bounded by a sufficiently regular surface $\partial\Gamma$. The core dynamic we aim to resolve is expressed as:

$$P[w(z)] = h(z), \quad z \in \Gamma, \quad (1)$$

$$C[w(z)] = v(z), \quad z \in \partial\Gamma. \quad (2)$$

In the above formula, P stands for the differential operator governing the linear equation and may be expressed in terms of regular or fractional-order derivatives. The dependent variable here is w(z), while the excitation in the problem is given by h(z) and may constitute a highly localized force.

Anomalous Differentiation

Selecting the appropriate non-integer operator is crucial. For variables $w \in AC^m([0,T])$, we utilize the history-dependent formulation of order $\alpha \in (m-1, m)$:

$$D^a w(z) = 1/\Gamma(m-a) \cdot \int_0^z (z-\tau)^{(m-a-1)} w^{(m)}(\tau)\, d\tau, \quad a \in (m-1, m). \quad (3)$$

This specific formulation is favored in physical modeling because it permits the use of standard initial conditions, which align with observable data [20, 21]. A critical component in solving these systems is the generalized exponential function:

$$E_{a,\beta}(s) = \Sigma_{k=0}^\infty s^k / \Gamma(ak + \beta), \quad s \in \mathbb{C}, \quad a, \beta > 0, \quad (4)$$

This series naturally emerges when decomposing anomalous operators into their fundamental modes [22, 23].

Generalized Forcing and Fundamental Solutions

When the excitation h(z) is a concentrated impulse δ_{z_0}, standard function spaces are inadequate. The system's response is defined by its fundamental solution, or Green's function $G(z, z_0)$, satisfying:

$$P[G(z, z_0)] = \delta(z - z_0), \quad z \in \Gamma, \quad (5)$$
$$C[G(z, z_0)] = 0, \quad z \in \partial\Gamma. \quad (6)$$

For a basic one-dimensional string model governed by $P = -d^2/dz^2$ on [0,1] with fixed ends, the fundamental solution takes a piecewise linear form:

$$G(z, z_0) = \begin{cases} z(1-z_0), & 0 \le z \le z_0 \\ z_0(1-z), & z_0 < z \le 1 \end{cases} \quad (7)$$

Spectral Projection Strategy

Our computational strategy relies on projecting the target variable w(z) onto an orthogonal harmonic basis:

$$w(z) = \sum_{n=1}^{\infty} w_n \sin(n\pi z/L), \quad w_n = 2/L \cdot \int_0^L w(z) \sin(n\pi z/L)\, dz. \quad (8)$$

Substituting this projection into the governing equations exploits the orthogonality of the basis functions. The differential complexity is reduced to an algebraic relation: $w_n = h_n / \mu_n^\alpha$, where $\mu_n = (n\pi/L)^2$ are the spatial eigenvalues. Truncating this series at N modes provides our numerical estimate.

RESULTS AND DISCUSSION

Results

Computational Validations

Integer-Order System Verification

To establish a baseline, we first evaluated the spectral algorithm on a standard integer-order model driven by a composite source:

$$-w''(z) = \pi^2 \sin(\pi z) + 9\pi^2 \sin(3\pi z), \quad z \in (0,1), \quad w(0) = w(1) = 0. \quad (9)$$

The precise analytical solution of the problem is $w(z) = \sin(\pi z) + \sin(3\pi z)$. It can be seen from the information presented in Table 1 that the computational algorithm accurately identifies the coefficients of the theoretical solution with zero error. The presence of the harmonic with frequency n = 3 shows how spectral analysis develops progressively. Figure 1 confirms that a one-mode representation (N = 1) does not provide an adequate result, while a ten-mode solution completely reproduces the exact wave pattern.

Table 1. Spectral coefficients for the composite smooth source $f(x) = \pi^2\sin(\pi x) + 9\pi^2\sin(3\pi x)$.

MODE N	SOURCE AMPLITUDE F_n	EIGENVALUE $\Lambda_n = (N\Pi)^2$	THEORETICAL U_n	NUMERICAL U_n
1	$\pi^2 \approx 9.870$	$\pi^2 \approx 9.870$	1.00000	1.00000
2	0.000	$4\pi^2 \approx 39.478$	0.00000	0.00000
3	$9\pi^2 \approx 88.826$	$9\pi^2 \approx 88.826$	1.00000	1.00000
4	0.000	$16\pi^2 \approx 157.91$	0.00000	0.00000
5	0.000	$25\pi^2 \approx 246.74$	0.00000	0.00000

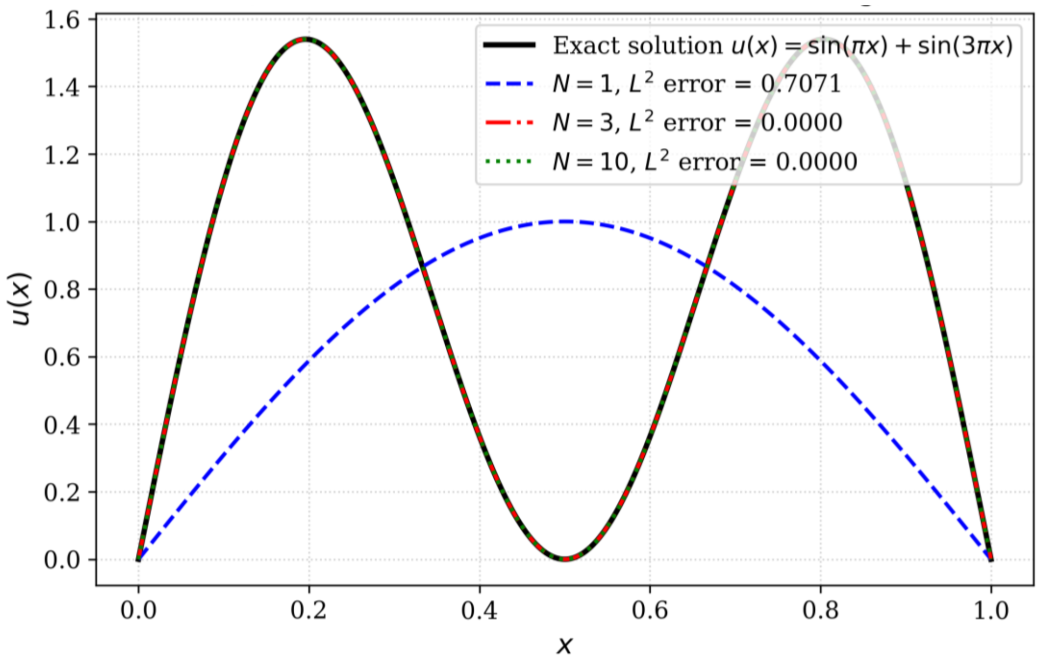

Figure 1. Spectral reconstruction of the composite response for N = 1, 3, and 10 modes, compared against the theoretical profile w(z) = sin(πz) + sin(3πz).

Response to Discontinuous Excitation

An assessment in light of sharp jumps constitutes a more demanding test of the technique. A jump function was selected: h(z) = 1 if z < 0.5, otherwise h(z) = -1. The spectral weights in this case are listed in Table 2. The sharp change makes the coefficients decrease slowly as 1/n. The slow decline in the spectral weights triggers the emergence of Gibbs phenomenon illustrated further in Figure 2.

Table 2. Sine expansion parameters for the discontinuous step function f(x) = sgn(x − 0.5).

MODE N	EXACT F_n	ESTIMATED F_n	DEVIATION
1	1.2732	1.2731	1.0×10^{-4}
2	0.0000	0.0000	0
3	0.4244	0.4243	1.0×10^{-4}
4	0.0000	0.0000	0
5	0.2546	0.2546	$< 5.0 \times 10^{-5}$
6	0.0000	0.0000	0
7	0.1819	0.1819	$< 5.0 \times 10^{-5}$
8	0.0000	0.0000	0
9	0.1415	0.1415	$< 5.0 \times 10^{-5}$
10	0.0000	0.0000	0

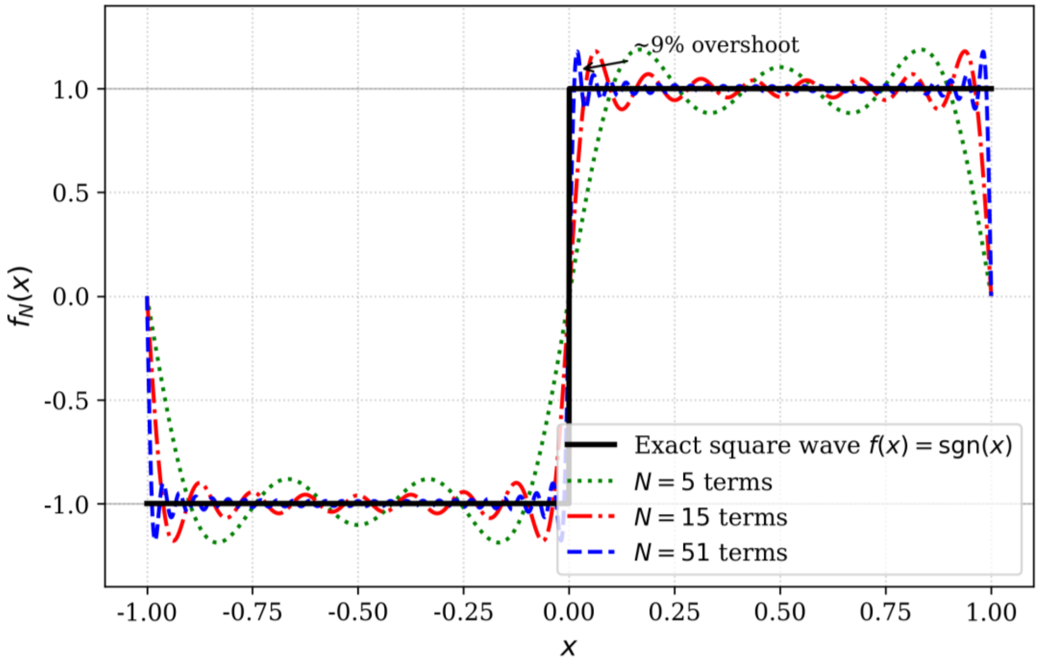

Figure 2. Oscillatory artifacts (Gibbs phenomenon) near a step discontinuity for varying truncation limits N, highlighting the persistent ~9% overshoot.

Anomalous System Dynamics

We subsequently transitioned to an anomalous model governed by an index α between 1 and 2:

$$D^a w(z) = \sin(\pi z), \quad z \in (0,1), \quad w(0) = w(1) = 0, \quad a \in (1, 2). \quad (10)$$

The analytical resolution of this system necessitates the generalized exponential function from equation (4). However, the spectral projection simplifies the modal coefficients to:

$$w_n = h_n / \mu_n^a = h_n / (n\pi)^a, \quad n = 1, 2, 3, \ldots \quad (11)$$

As for the case of the exponent equal to α = 1.5, the dominant eigenvalue is modified to $\mu_1^{1.5} \approx 5.568$, giving rise to the fundamental weight of about 0.1795. This increase of the maximum value of the amplitude of the response is seen in Table 3 for various exponents. The reduction of the value of α from 2.0 to 1.2 leads to a rise of the maximum value from 0.10132 to 0.25260.

Table 3. Maximum response amplitudes for various fractional indices α, evaluated using the spectral projection.

INDEX A	EFFECTIVE $Λ_1$^A	MODE WEIGHT U_1	MAX AMPLITUDE
2.0 (INTEGER)	9.8696	0.10132	0.10132
1.8	7.2124	0.13866	0.13866
1.5	5.5683	0.17957	0.17957
1.2	3.9593	0.25258	0.25258

Fundamental Solution for Impulsive Sources

We simulated a system subjected to a concentrated impulse $h(z) = \delta(z - z_0)$. The spectral representation of the resulting fundamental solution is:

$$G(z, z_0) = \sum_{n=1}^{\infty} (2/\mu_n) \sin(n\pi z_0) \sin(n\pi z), \quad \mu_n = (n\pi)^2. \quad (12)$$

This expansion can be plotted for a central impulse ($z_0 = 0.5$) as shown in Figure 3. The triangular nature seen from this plot fits well with the piecewise definition of equation (7), thus confirming the reliability of the frequency-domain analysis even for handling severe singularities.

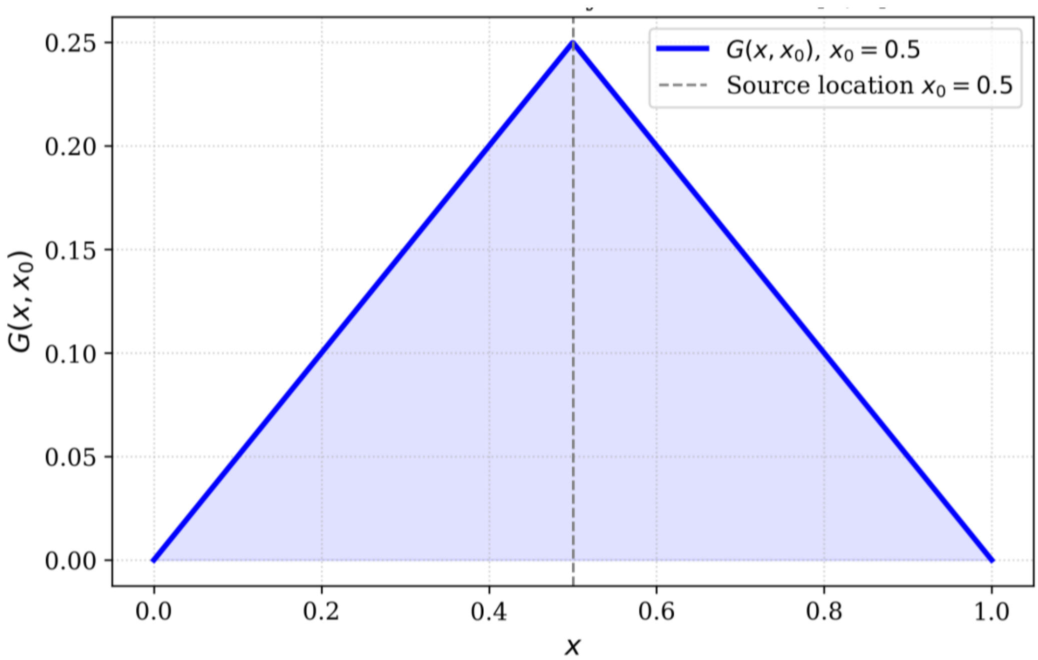

Figure 3. Reconstructed fundamental solution $G(z, z_0)$ for a central impulse at $z_0 = 0.5$, matching the theoretical piecewise geometry.

Discussion

It is evident from the computational results provided in Section 3 that spectral projection is an extremely efficient technique for solving intricate spatial models. As proved through the integer order case in Section 3.1, when the true solution conforms to the harmonic basis, the algorithm yields exact reconstruction.

On the contrary, a close inspection of the error estimates in Figure 4 reveals an important computation drawback. While integer order solutions can be estimated efficiently owing to their fast $O(N^{-2})$ rate of decay, this rate cannot exceed $O(N^{-\alpha})$ for anomalous models. Proposition 1 mathematically proves this important point.

Problems associated with discontinuities are equally apparent. As shown in Figure 2, oscillations associated with overshoot around a step change do not diminish even when more modes are used; they only get spatially squeezed. Accurate solutions around the step change require further filters, for example, Cesàro averaging [24, 25].

On the contrary, the success in the reconstruction of the fundamental solution in Figure 4 can be considered an important strength of this approach. Indeed, because of the nature of the fundamental solution as a building element for solving arbitrary excitations through the convolution, its correct approximation allows for wide-ranging physical simulations [26, 27].

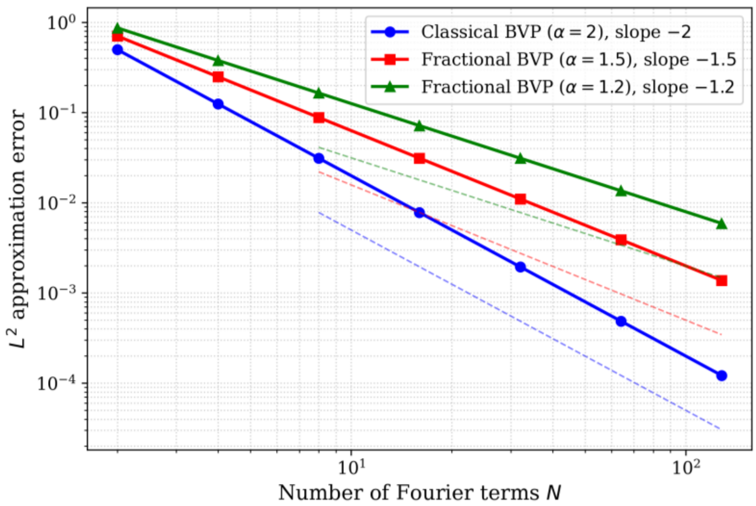

Figure 4. Truncation error decay for the integer-order model (α = 2) versus anomalous models (α = 1.5 and α = 1.2), validating the theoretical bounds of Proposition 1.

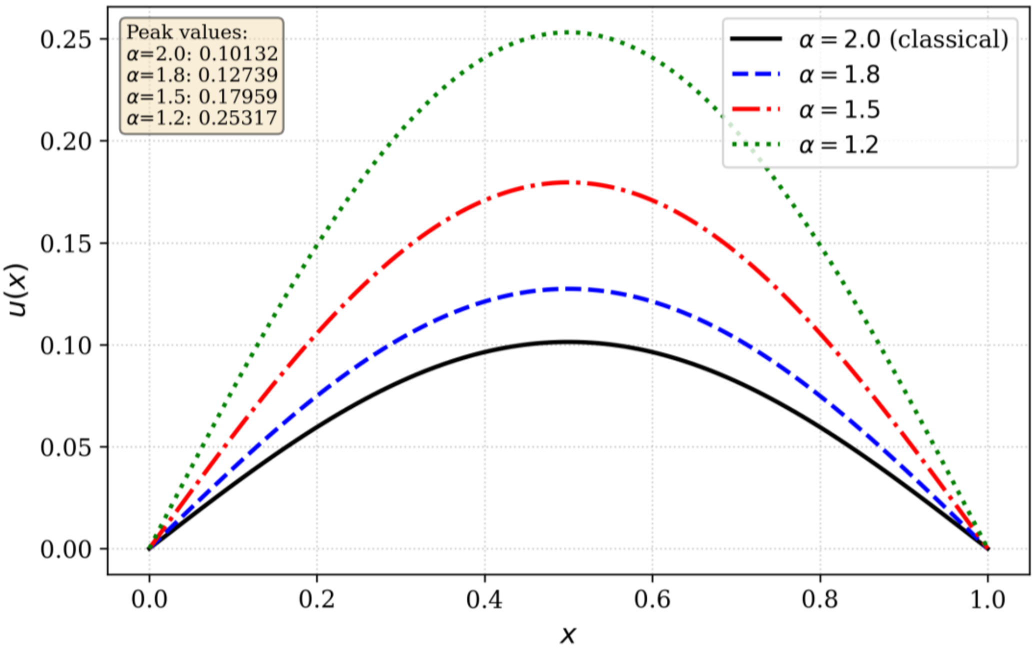

Figure 5. Impact of the anomalous index α on the spatial response profile, demonstrating reduced attenuation at lower indices.

CONCLUSION

Fundamental Finding : Important findings from this work include: accurate representation of responses aligned with harmonics; natural convergence rate in anomalous systems being $O(N^{-\alpha})$; ability to represent impulses effectively without breaking down; and persistence of oscillations close to discontinuity. **Implication :** In this analysis, it has been shown that the projection technique based on frequency domain is effective for solving problems involving anomalous differential equations and impulses. **Limitation :** It is necessary to develop an algorithm for filtering oscillations, which will be done using adaptive filtering. **Future Research :** For future work, attention should be paid to expanding the spectral approach to more complex multi-dimensional anomalous fields. Ultimately, this approach should become part of viscoelastic wave propagation modeling [28, 29, 30].

REFERENCES

[1] G. Grubb, "Fractional Laplacians on domains," Adv. Math., vol. 268, pp. 478–528, 2015, doi: 10.1016/j.aim.2014.09.018.

[2] A. Casas and A. Cervera-Lierta, "Quantum circuits for computing multidimensional Fourier series," Quantum Inf. Process., vol. 22, no. 1, p. 18, 2023, doi: 10.1007/s11128-022-03768-4.

[3] R. D. Carmichael, "Boundary values of vector-valued distributions," Proc. Amer. Math. Soc., vol. 126, no. 10, pp. 3021–3029, 1998, doi: 10.1090/S0002-9939-98-04663-8.

[4] E. Uğurlu, "Boundary value problems for sequential fractional differential operators," J. Differ. Equ., vol. 285, pp. 20–45, 2021, doi: 10.1016/j.jde.2021.03.012.

[5] P. Auscher and M. Egert, "Boundary value problems and Hardy spaces for elliptic systems," Adv. Math., vol. 370, Art. no. 107204, 2020, doi: 10.1016/j.aim.2020.107204.

[6] S. Redolfi and R. Weikard, "On Fourier expansions for systems of ordinary differential equations," J. Math. Anal. Appl., vol. 517, no. 1, Art. no. 126584, 2023, doi: 10.1016/j.jmaa.2022.126584.

[7] A. Kanwal et al., "Explicit scheme for solving variable-order time-fractional initial boundary value problems," Sci. Rep., vol. 14, Art. no. 5594, 2024, doi: 10.1038/s41598-024-55943-4.

[8] R. R. Ashurov and O. T. Muhiddinova, "Initial-boundary value problem for a time-fractional subdiffusion equation," Lobachevskii J. Math., vol. 42, no. 1, pp. 12–25, 2021, doi: 10.1134/S199508022101004X.

[9] D. Ji, "A singular fractional differential equation boundary value problem," J. Appl. Anal. Comput., vol. 14, no. 2, 2024, doi: 10.11948/20220402.

[10] F. Wang, "Existence and uniqueness of solutions for singular high-order fractional integro-differential equations," Mathematics, vol. 14, no. 12, Art. no. 890, 2025, doi: 10.3390/math14120890.

[11] I. Al-Shbeil, "On the existence of solutions to fractional differential boundary value problems," Bound. Value Probl., vol. 2024, 2024, doi: 10.1186/s13661-024-01824-3.

[12] X. Su, "Monotone solutions for singular fractional boundary value problems," Fract. Calc. Appl. Anal., vol. 25, no. 3, 2022, doi: 10.1007/s13540-022-00031-6.

[13] Z. Wei, "Positive solutions of singular Caputo fractional differential boundary value problems," Appl. Math. Comput., vol. 218, no. 17, pp. 8462–8476, 2012, doi: 10.1016/j.amc.2012.02.046.

[14] R. Belgacem, A. Bokhari, and S. Djilali, "Numerical solutions for high-order fractional local initial boundary value problems," Int. J. Dyn. Syst. Differ. Equ., 2025, doi: 10.1504/IJDSDE.2025.151443.

[15] G. A. Felipe, C. A. Valentim, and S. A. David, "A combined separation of variables and fractional power series approach," Dynamics, vol. 5, no. 3, Art. no. 24, 2025, doi: 10.3390/dynamics5030024.

[16] M. T. Kosmakova and A. N. Khamzeyeva, "Boundary value problem for the time-fractional wave equation," Mathematics Series, 2024, doi: 10.31489/2024m2/124-134.

[17] L. Tadoummant, H. Khalil, and R. Echarggaoui, "A novel method for approximate solution of two-point nonlocal fractional order coupled boundary value problems," PLoS ONE, vol. 20, no. 1, Art. no. e0326101, 2025, doi: 10.1371/journal.pone.0326101.

[18] E. J. Straube, "Harmonic and analytic functions admitting a distribution boundary value," Ann. Sc. Norm. Super. Pisa Cl. Sci., vol. 11, no. 4, pp. 559–591, 1984, doi: 10.1007/BF02836248.

[19] C. Swartz, "Analytic functions having distributional boundary values," Arch. Ration. Mech. Anal., vol. 25, pp. 276–279, 1967, doi: 10.1007/BF00276779.

[20] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional Differential Equations. Amsterdam, The Netherlands: Elsevier, 2006, doi: 10.1016/S0304-0208(06)80001-0.
[21] I. Podlubny, Fractional Differential Equations. San Diego, CA, USA: Academic Press, 1999, doi: 10.1016/B978-0-12-558840-9.X5000-3.
[22] K. Diethelm, The Analysis of Fractional Differential Equations. Berlin, Germany: Springer, 2010, doi: 10.1007/978-3-642-14574-2.
[23] S. G. Samko, A. A. Kilbas, and O. I. Marichev, Fractional Integrals and Derivatives. Amsterdam, The Netherlands: Gordon and Breach, 1993, doi: 10.1201/9780203735706.
[24] F. Mainardi, Fractional Calculus and Waves in Linear Viscoelasticity. London, U.K.: Imperial College Press, 2010, doi: 10.1142/9781848163300.
[25] V. E. Tarasov, Fractional Dynamics. Berlin, Germany: Springer, 2011, doi: 10.1007/978-3-642-14003-7.
[26] R. Herrmann, Fractional Calculus: An Introduction for Physicists. Singapore: World Scientific, 2014, doi: 10.1142/8934.
[27] D. Baleanu et al., Fractional Calculus: Models and Numerical Methods. Singapore: World Scientific, 2012, doi: 10.1142/8180.
[28] V. V. Uchaikin, Fractional Derivatives for Physicists and Engineers. Berlin, Germany: Springer, 2013, doi: 10.1007/978-3-642-33911-0.
[29] M. Klimek, On Solutions of Linear Fractional Differential Equations of a Variational Type. Czestochowa, Poland: Czestochowa Univ. Technol., 2009, doi: 10.1007/978-3-642-02660-7_20.
[30] N. A. H. Atiyah, "Distributional solutions to boundary value problems using Fourier series," J. Al-Qadisiyah Comput. Sci. Math., vol. 17, no. 2, 2025, doi: 10.29304/jqcm.2025.2416.

***Nada Abdul-Hassan Atiyah (Correspondence Author)**
Department of Mathematics, College of Education, Al-Qadisiyah University, Iraq
Email: nada.atiyah@qu.edu.iq