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Hilfer Kasr Tartibli Operatorni O‘Z Ichiga Oluvchi Korteveg-De Vries-Benjamin-Bona-Mahoni Tenglamasi Uchun Chegaraviy Masala Yechimining Mavjudligi Haqida

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Annotatsiya: Ushbu maqolada vaqt bo‘yicha kasr tartibli Korteveg-de Vries-Benjamin-Bona-Mahoni tenglamasi uchun chegaraviy masalaning yechim mavjud emasligi shartlari o‘rganilgan. Hilfer hosilalari asosida kasr tartibli operatorlar tahlil qilingan va yechim mavjud emasligini ko‘rsatish uchun Poxojayev usulidan foydalanilgan. Shuningdek, masalaning yechim mavjud emasligini isbotlovchi teorema va misollar keltirilgan.

Kalit so‘zlar: Korteveg-de Vries-Benjamin-Bona-Mahoni tenglamasi, kasr tartibli hosilalar, Hilfer hosilasi, Poxojayev usuli, yechim mavjud emasligi, vaqt bo‘yicha kasr tartibli tenglamalar.

1. Kirish

Korteveg-de Vries-Benjamin-Bona-Mahoni tenglamasi xussusiy hosilali differensial tenglama bo'lib, suyuqlik oqimining harakatini tavsiflaydi. Klassik Korteveg-de Vries-Benjamin-Bona-Mahoni tenglamasi quyidagi ko‘rinishda bo‘ladi:

$$u_t(t,x)-u_{txx}(t,x)+u_{xxx}(t,x)+uu_x(t,x)=0$$

bu yerda $u(t,x)$ – suyuqlikning tezligi, t – vaqt, x – pozitsiya va ν - suyuqlikning kinematik yopishqoqligi. Korteveg-de Vries-Benjamin-Bona-Mahoni tenglamasi chiziqli bo‘lgan tenglama bo‘lib, u suyuqliklar mexanikasi, transport oqimi va matematik-fizika kabi ko‘plab sohalarda qo‘llaniladi[5].

Mazkur maqolada biz kasr tartibli differensial operatorlarni o‘z ichiga oluvchi Korteveg-de Vries-Benjamin-Bona-Mahoni tenglamasi uchun chegaraviy masala yechimining mavjud bo‘lmaslik shartlarini aniqlash bilan shug‘ullanamiz.

2. Asosiy qism

1. Dastlab, biz kasr hisobning ba’zi asosiy tushuncha va ta’riflarini keltiraylik.

1.1-ta’rif. $f\in L([a,b])$ va $\alpha>0$ bo‘lsin. U holda quyidagi

$$I_{a+}^{\alpha}[f](t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds \tag{1.1}$$

va

$$I_{b-}^{\alpha}[f](t)=\frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}f(s)ds \tag{1.2}$$

integrallar mos holda Riman - Liuvillning chap tomonli va o‘ng tomonli kasr tartibli integrallari deb ataladi, bu yerda $\Gamma(z)$ – Eylerning gamma - funksiyasi.

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1.2-ta'rif. Riman - Liuvillning chap tomonli $\alpha\left(0<\alpha<1\right)$ kasr tartibli hosilasi $D_{a+}^{\alpha}f$ quyidagicha aniqlanadi :

$$D_{a+}^{\alpha}[f](t)=\frac{d}{dt}I_{a+}^{1-\alpha}[f](t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{a}^{t}(t-s)^{-\alpha}f(s)ds. \qquad (1.3)$$

1.3-ta'rif. Riman - Liuvillning o'ng tomonli $\alpha\left(0<\alpha<1\right)$ kasr tartibli hosilasi $D_{b-}^{\alpha}f$ quyidagicha aniqlanadi :

$$D_{b-}^{\alpha}[f](t)=-\frac{d}{dt}I_{b-}^{1-\alpha}[f](t)=-\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{t}^{b}(s-t)^{-\alpha}f(s)ds. \qquad (1.4)$$

1.4-ta'rif. Kaputoning $\alpha\left(n-1<\alpha<n\right)$ kasr nartibli hosilasi quyidagicha aniqlanadi:

$$ {}_{a}^{C}D_{t}^{\alpha}f(t)=\frac{1}{\Gamma(\alpha-n)}\int_{a}^{t}\frac{f^{n}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau \qquad (1.5)$$

1.5-ta'rif. Hilferning $0<\alpha<1$ tartibli va $0\le\beta\le1$ tipli hosilasi $D_{a+}^{\alpha,\beta}f$ quyidagicha aniqlanadi:

$$D_{a+}^{\alpha,\beta}[f](t)=I_{a+}^{\beta(1-\alpha)}\frac{d}{dt}I_{a+}^{(1-\beta)(1-\alpha)}[f](t), \qquad (1.6)$$

bu yerda I_{a+}^{σ}, $\sigma>0$ – Riman - Liuvill kasr tartibli integrali.

Hilfer hosilasi haqida batafsil ma'lumot uchun [2] , [3] ishlarni tavsiya etamiz. Bu hosila $\left(\beta=0\right)$ da Riman - Liuvill hosilasi va $\left(\beta=1\right)$ da Kaputo kasr hosilasiga aylanadi [1].

1.1-lemma. $\alpha>0, p\ge1, q\ge1$ va $\frac{1}{p}+\frac{1}{q}\le1+\alpha$ ($\frac{1}{p}+\frac{1}{q}=1+\alpha$ holatda $p\ne1$ va $q\ne1$) bo'lsin. Agar $\varphi\in L_{p}(a,b)$ va $\psi\in L_{q}(a,b)$ bo'lsa, u holda

$$\int_{a}^{b}\varphi(t)I_{a+}^{\alpha}[\psi](t)dt=\int_{a}^{b}\psi(t)I_{b-}^{a}[\varphi]dt \qquad (1.7)$$

tenglik o'rinli.

2. Vaqt bo'yicha kasr tartibli Korteveg-de Vries-Benjamin-Bona-Mahoni tenglamasi yechimining mavjud emasligi.

$\mathbf{R}^{2}$ dagi to'g'ri to'rtburchakli sohani $\Pi_{a,b}$ bilan belgilaylik:

$$\Pi_{a,b}=\left\{(t,x)\in\mathbf{R}^{2}:0<t<T, a<x<b\right\}.$$

$\Pi_{a,b}$ sohada vaqt bo'yicha kasr tartibli ushbu

$$D_{0+,t}^{\alpha,\beta}u(t,x)-u_{txx}(t,x)+u_{xxx}+u(t,x)u_{x}(t,x)=0 \qquad (2.1)$$

Korteveg-de Vries-Benjamin-Bona-Mahoni tenglamasi va quyidagi boshlang'ich shart

$$I_{0+,t}^{\gamma-1}u(0,x)=u_{0}(x), x\in[a,b] \qquad (2.2)$$

berilgan bo‘lsin, bu yerda $D_{a+}^{\alpha,\beta}$ – $0<\alpha<1$ tartibli va $0\le\beta\le1$ tipli Hilfer hosilasi, $u_0(x)$ berilgan funksiya va $\gamma=(1-\beta)(1-\alpha)+1$ ga teng bo‘lgan haqiqiy son.

(2.1) - (2.2) masala yechimining mavjudligi masalasini qaraylik. Buning uchun nochiziqli tenglamalar yechimlarining buzilishini tahlil qilish uchun Poxojayev tomonidan taklif etilgan usuldan foydalanamiz [6].

$T>0$, $a,b\in\mathbf{R}$ ixtiyoriy parametrlar bilan $\Pi_{a,b}$ sohada aniqlangan $\varphi(t,x)$ funksiyalarning $\Phi(\Pi_{a,b})$ sinfini qaraylik. Bu sinf funksiyalari quyidagi xossalarga ega:

(i) $\varphi_t,\varphi_{xx}\in C(\Pi_{a,b})$;

(ii) $\Pi_{a,b}$ da $\varphi_x\ge0$;

(iii) $x\in(a,b)$ va $t=T$ da $I_{T-,t}^{\beta(1-\alpha)}\varphi(x,t)=0$;

(iv) $\zeta(\Pi_{a,b})=\iint\limits_{\Pi_{a,b}}\frac{(L^*\varphi)^2}{\varphi_x}dtdx<+\infty$;

bu yerda $L^*\varphi=-I_{T-,t}^{(1-\beta)(1-\alpha)}D_{T-,t}^{1-\beta(1-\alpha)}\varphi-\varphi_{xxt}+\varphi_{xxx}$.

Faraz qilaylik, (2.1) - (2.2) masalaning ixtiyoriy $\varphi(x,t)\in\Phi(\Pi_{a,b})$ uchun $u_{xx},D_{0+,t}^{\alpha,\beta}u\in C([a,b]\times[0,t])$ shartni qanoatlantiradigan sust yechimi va $T>0$ son mavjud bo‘lsin.

Endi (2.1) tenglamani $\varphi\in\Phi(\Pi_{a,b})$ funksiyaga ko‘paytirib, so‘ngra $\Pi_{a,b}$ bo‘yicha integrallab, quyidagiga ega bo‘lamiz:

$$\iint\limits_{\Pi_{a,b}}\varphi(t,x)D_{0+,t}^{\alpha,\beta}u(t,x)dtdx-\iint\limits_{\Pi_{a,b}}\varphi(t,x)u_{txx}(t,x)dtdx+\iint\limits_{\Pi_{a,b}}\varphi(t,x)u_{xxx}(t,x)dtdx+$$
$$+\iint\limits_{\Pi_{a,b}}\varphi(t,x)u(t,x)u_x(t,x)dtdx=0. \qquad (2.3)$$

(2.3) tenglikdagi integrallarda bo‘laklab integrallash qoidasidan foydalanib, quyidagi tengliklarni hosil qilamiz:

$$\iint\limits_{\Pi_{a,b}}\varphi(t,x)u(t,x)u_x(t,x)dtdx=$$
$$=\frac{1}{2}\int_0^T u^2(t,x)\varphi(t,x)\Big|_a^b dt-\frac{1}{2}\iint\limits_{\Pi_{a,b}}u^2(t,x)\varphi_x(t,x)dtdx, \qquad (2.4)$$

$$\iint\limits_{\Pi_{a,b}}\varphi(t,x)u_{xxx}(t,x)dtdx==\int_0^T\left[u_{xx}(t,x)\varphi(t,x)-u_x(t,x)\varphi_x(t,x)-u(t,x)\varphi_{xx}(t,x)\right]\Big|_a^b dt+$$
$$+\iint\limits_{\Pi_{a,b}}u(t,x)\varphi_{xxx}(t,x)dtdx. \qquad (2.5)$$

$$\iint_{\Pi_{a,b}} \varphi(t,x)u_{txx}(t,x)dtdx =$$

$$= \int_0^T \left[u_{tx}(t,x)\varphi(t,x) - u_t(t,x)\varphi_x(t,x)\right]\Big|_a^b dt - \int_a^b u(t,x)\varphi_{xx}(t,x)\Big|_0^T dx +$$

$$+ \iint_{\Pi_{a,b}} u(t,x)\varphi_{xxt}(t,x)dtdx. \qquad (2.6)$$

1.5-ta'rif hamda 1.1-lemmadan foydalansak, u holda quyidagi tenglik o'rinli:

$$\iint_{\Pi_{a,b}} \varphi(t,x)D_{0+,t}^{\alpha,\beta}u(t,x)dtdx = \iint_{\Pi_{a,b}} \varphi(t,x)I_{0+}^{\beta(1-\alpha)}\frac{d}{dt}I_{0+}^{(1-\beta)(1-\alpha)}u(t,x)dtdx =$$

$$= \iint_{\Pi_{a,b}} I_{T-,t}^{\beta(1-\alpha)}\varphi(t,x)\frac{d}{dt}I_{0+}^{(1-\beta)(1-\alpha)}u(t,x)dtdx.$$

Oxirgi tenglikda bo'laklab integrallash qoidasini qo'llab va 1.1-lemmadan yana bir marta foydalanib, quyidagi tenglikka ega bo'lamiz:

$$\iint_{\Pi_{a,b}} \varphi(t,x)D_{0+,t}^{\alpha,\beta}u(t,x)dtdx = \int_a^b \left\{I_{0+,t}^{(1-\beta)(1-\alpha)}u(t,x)I_{T-,t}^{\beta(1-\alpha)}\varphi(t,x)\right\}\Big|_0^T dx -$$

$$- \iint_{\Pi_{a,b}} \frac{d}{dt}I_{T-,t}^{\beta(1-\alpha)}\varphi(t,x)I_{0+}^{(1-\beta)(1-\alpha)}u(t,x)dtdx =$$

$$\int_a^b \left\{I_{0+,t}^{(1-\beta)(1-\alpha)}u(t,x)I_{T-,t}^{\beta(1-\alpha)}\varphi(t,x)\right\}\Big|_0^T dx - \iint_{\Pi_{a,b}} u(t,x)I_{T-,t}^{(1-\beta)(1-\alpha)}\frac{d}{dt}I_{T-,t}^{\beta(1-\alpha)}\varphi(t,x)dtdx.$$

(2.4) va (2.5) dan va 1.3-ta'rifdan foydalansak, (2.3) quyidagi ko'rinishni oladi:

$$\frac{1}{2}\iint_{\Pi_{a,b}} u^2(t,x)\varphi_x(t,x)dtdx = \iint_{\Pi_{a,b}} u(t,x)\left(L^*\varphi\right)(t,x) +$$

$$+ \int_a^b \left\{I_{0+,t}^{(1-\beta)(1-\alpha)}u(t,x)I_{T-,t}^{\beta(1-\alpha)}\varphi(t,x) + u(t,x)\varphi_{xx}(t,x)\right\}\Big|_0^T dx +$$

$$+ \int_0^T B(u(t,x),\varphi(t,x))\Big|_a^b dt \qquad (2.7)$$

bu yerda

$$B(u(t,x),\varphi(t,x)) = \frac{1}{2}u^2(t,x)\varphi(t,x) - u_{tx}(t,x)\varphi(t,x) + u_t(t,x)\varphi_x(t,x) +$$

$$u_{xx}(t,x)\varphi(t,x) - u_x(t,x)\varphi_x(t,x) - u(t,x)\varphi_{xx}(t,x).$$

(2.2) va $\varphi(t,x)$ funksiyaning *(iii)* xossasidan foydalansak, quyidagilarni topamiz:

$$\frac{1}{2}\iint_{\Pi_{a,b}} u^2(t,x)\varphi_x(t,x)dtdx = \iint_{\Pi_{a,b}} u(t,x)\left(L^*\varphi\right)(t,x)dtdx + \int_0^T B\left(u(t,x),\varphi(t,x)\right)\Big|_a^b dt -$$

$$-\int_a^b (u_0(x) I_{T-,t}^{\beta(1-\alpha)}\varphi(x,t)\big|_{t=0}) + (u(t,x)\varphi_{xx}(x,t))\big|_0^T)dx. \qquad (2.8)$$

Gyolder va Yung tengsizliklaridan foydalansak,

$$\left|\iint_{\Pi_{a,b}} u(t,x)\left(L^*\varphi\right)(t,x)dtdx\right| = \left|\iint_{\Pi_{a,b}} u(t,x)\sqrt{\varphi_x(x,t)}\frac{\left(L^*\varphi\right)(t,x)}{\sqrt{\varphi_x(x,t)}}dtdx\right| \le$$

$$\left(\iint_{\Pi_{a,b}} u^2(t,x)\varphi_x(t,x)dtdx\right)^{1/2}\left(\iint_{\Pi_{a,b}}\frac{\left(\left(L^*\varphi\right)(t,x)\right)^2}{\varphi_x(x,t)}dtdx\right)^{1/2} \le$$

$$\frac{1}{2}\iint_{\Pi_{a,b}} u^2(t,x)\varphi_x(t,x)dtdx + \frac{1}{2}\iint_{\Pi_{a,b}}\frac{\left(\left(L^*\varphi\right)(t,x)\right)^2}{\varphi_x(x,t)}dtdx$$

munosbatga ega bo‘lamiz.

Bu tengsizlikni va *(iv)* xossani e’tiborga olsak, (2.8) quyidagicha ko‘rinish oladi:

$$0 \le \frac{1}{2}\zeta\left(\Pi_{a,b}\right) + \int_0^T B\left(u(t,x),\varphi(t,x)\right)\Big|_a^b dt - .$$

$$-\int_a^b (u_0(x) I_{T-,t}^{\beta(1-\alpha)}\varphi(x,t)\big|_{t=0}) + (u(t,x)\varphi_{xx}(x,t))\big|_0^T)dx. \qquad (2.9)$$

Quyidagi teorema o‘rinli:

2.1-teorema. Faraz qilaylik, $u_0(x) \in L[a,b]$ bo‘lib, $u_0(x)$ va chegaraviy shartlarda berilgan funksiyalar ushbu shartlarni qanoatlantirsin: shunday $\varphi(x,t) \in \Phi\left(\Pi_{a,b}\right)$ mavjudki, $B\left(u(t,x),\varphi(t,x)\right)\Big|_a^b \in L[0,T]$ bo‘lib, quyidagi tengsizlik o‘rinli bo‘lsin:

$$\frac{1}{2}\zeta\left(\Pi_{a,b}\right) + \int_0^T B\left(u(t,x),\varphi(t,x)\right)\Big|_a^b dt - .$$

$$-\int_a^b (u_0(x) I_{T-,t}^{\beta(1-\alpha)}\varphi(x,t)\big|_{t=0}) + (u(t,x)\varphi_{xx}(x,t))\big|_0^T)dx < 0 \qquad (2.10)$$

U holda (2.1) - (2.2) masala $u_{xx}, D_{0+,t}^{\alpha,\beta}u \in C\left([a,b]\times[0,t]\right)$ yechimi $\Pi_{a,b}$ da mavjud bo‘lmaydi.

Isbot. Teskarisidan faraz qilaylik, ya’ni (2.1) – (2.2) masala $\Pi_{a,b}$ da yechimga ega bo‘lsin. U holda (2.9) va (2.10) tengsizliklarga ko‘ra qarama-qarshilikka duch kelamiz. Bundan esa farazimiz noto‘g‘ri ekanligi kelib chiqadi.

Misol sifatida $\Pi_{a,b}=\left\{(t,x)\in\mathbf{R}^2:0<t<T,0<x<1\right\}$ to‘g‘ri to‘rtburchakli sohada (2.1) Korteveg-de Vries-Benjamin-Bona-Mahoni kasr tartibli tenglamasini (2.2) boshlang‘ich shart va

$$u(t,0)=\tau_1(t),\ u_x(t,0)=\tau_2(t),\ u_{xx}(t,0)=\tau_3(t),\ u_{tx}(t,0)=\tau_4(t),\ u_t(t,0)=\tau_5(t),$$
$$0<t<T \qquad (2.11)$$

chegaraviy shartlar bilan qaraylik, bu yerda $\tau_1,\tau_2,\tau_3,\tau_4$ va τ_5 berilgan funksiyalar bo‘lib, $\tau_1,\tau_2,\tau_3,\tau_4,\tau_5\in L[0,T]$.

(2.1) vaqt bo‘yicha kasr tartibli Korteveg-de Vries-Benjamin-Bona-Mahoni tenglamasini $\varphi\in\Phi(\Pi_{a,b})$ ga ko‘paytirganimizdan keying hisoblashlar va soddalashtirishlardan so‘ng quyidagiga kelamiz:

$$0<\frac{1}{2}\zeta(\Pi_{0,1})+\int_0^T B(u(t,x),\varphi(t,x))\Big|_a^b dt-$$
$$-\int_0^1 (u_0(x)I_{T-,t}^{\beta(1-\alpha)}\varphi(x,t)\Big|_{t=0})+(u(t,x)\varphi_{xx}(x,t))\Big|_0^T)dx.$$

$\varphi(t,x)$ funksiyaga chegaraviy shartlarni

$$\varphi(t,1)=0,\ \varphi_x(t,1)=0,\ 0<t<T. \qquad (2.12)$$

kabi qo‘yaylik.

U holda quyidagiga ega bo‘lamiz:

$$B(u,\varphi)\Big|_a^b==-\left[\frac{1}{2}\tau_1^2(t)+\tau_3(t)-\tau_4(t)\right]\varphi(t,0)+(\tau_2(t)-\tau_5(t))\varphi_x(t,0)-$$
$$-\tau_1(t))\varphi_{xx}(t,0).$$

Bu holda quyidagi teorema o‘rinli:

2.2-teoreama. (2.1), (2.2), (2.11) boshlang‘ich chegaraviy masalaning (2.12) chegaraviy shartlarni qanoatlantiruvchi $\varphi\in\Phi(\Pi_{0,1})$ funksiyasi mavjud bo‘lib, quyidagi tengsizlik o‘rinli bo‘lsin:

$$\frac{1}{2}\zeta(\Pi_{0,1})<\int_0^T(\frac{1}{2}\tau_1^2(t)\varphi(t,0)+\tau_3(t)\varphi(t,0)-\tau_4(t)\varphi(t,0)-\tau_2(t)\varphi_x(t,0)+$$
$$+\tau_5(t)\varphi_x(t,0)+\tau_1(t)\varphi_{xx}(t,0))dx+$$
$$+\int_0^1(u_0(x)I_{T-,t}^{\beta(1-\alpha)}\varphi(x,t)\Big|_{t=0})+(u(t,x)\varphi_{xx}(x,t))\Big|_0^T)dx. \qquad (2.13)$$

U holda (2.1), (2.2), (2.11) masala $\Pi_{0,1}$ da yechimga ega bo‘lmaydi.

2.1-misol. $\varphi(t,x)$ funksiyasi sifatida quyidagi funksiyani olishimiz mumkin:

$$\varphi(t,x)=(T-t)^3(x-1)^3, \qquad (2.14)$$

(2.13) funksiya uchun *(i)-(iv)* shartlar bajarilishini osongina ko‘rsatish mumkin. Va shuningdek bu funksiyadan foydalanib, osongina quyidagi tenglikni ko‘rsatamiz:

$$\zeta\left(\Pi_{0,1}\right)=\frac{k_1^2}{3}\frac{T^{4-2\alpha}}{4-2\alpha}-4k_1\frac{T^{4-\alpha}}{4-\alpha}+9T^4, \qquad (2.15)$$

bu yerda $k_1=\Gamma(4)\Gamma^{-1}(4-\alpha)$.

2.2- teoremadan quyidagi natijaga kelamiz:

Natija. $u_0\in L[0,1]$, $\tau_1,\tau_2\in L[0,T]$ funksiyalar quyidagi tengsizlikni qanoatlantirsin:

$$k_2\int_0^1 u_0(x)(x-1)^3\,dx>\int_0^T\left[\frac{1}{2}\tau_1^2(t)+\tau_3(t)-\tau_4(t)+3\tau_2(t)-3\tau_5(t)+6\tau_1(t)\right](T-t)^3\,dt+$$

$$\frac{k_1^2}{6}\frac{T^{4-2\alpha}}{4-2\alpha}-2k_1\frac{T^{4-\alpha}}{4-\alpha}+9\frac{T^4}{2}.$$

bu yerda $k_2=\Gamma(4)\Gamma^{-1}(\beta(1-\alpha)+4)$.

U holda (2.1), (2.2), (2.11) masala $\Pi_{0,1}$ da yechimga ega bo‘lmaydi.

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