Journal of Theory, Mathematics and Physics
Vol. 3, No. 10, 2024 ISSN: 2181-4376

FINDING THE AREA OF TRIANGLES USING GALILEAN GEOMETRY ELEMENTS

Annotation: *In this article, various problems of finding the areas of triangles using the elements of Galilean geometry related to semi-Euclidean space geometry are presented and proved. Although concepts from Euclidean geometry are used in the concepts in this article, their geometric meanings are fundamentally different.*

Keywords: *triangle, area of a triangle, angle, height, parabola.*

Information about the authors

Sultanov Bekzod, Obidjanov Islombek, Shamuratov Elyor, Sobirov Hojiakbar

Department of Algebra and Mathematical Engineering, Urganch State University;

Introduction.

Teaching mathematics particualarly geometry, in schools is of great impordonce in fostering a culture of thingking. Euclidean geometry is not taught in schools, thought, we use elements of Galileon geometry to find the area of a triangle in order to develop geometric imagination of students.

Preliminaries

First, let us introduce the basic concepts of Galilean geometry. Let the vectors $\vec{X}(x_1, y_1)$ and $\vec{Y}(x_2, y_2)$ be given in the affine space A_2.

Definition : The affine space A_2 is colled the R_2^1 Galilean plane, if the inner product of the vectors $\vec{X}(x_1, y_1)$ and $\vec{Y}(x_2, y_2)$ is defined as

$$\begin{cases} (XY)_1 = x_1 x_2 \quad , & if \quad X_1 X_2 \neq 0 \\ (XY)_2 = y_1 y_2 \quad , & if \quad (XY)_1 = 0 \end{cases}$$

for all $A_2 \ni X, Y$.

The norm of a vector in space R_2^1 is equal to the square root of the scolar product of this vector on it self.

$$\|\vec{X}\| = \begin{cases} x_1 \quad qachonki \quad x_1 \neq 0 \\ y_1 \quad qachonki \quad x_1 = 0. \end{cases}$$

The distance between two points in R_2^1 $A(x_1,y_1)$ and $B(x_2,y_2)$ is the norm of the vector connecting the two point:

$$|\overrightarrow{AB}| = \begin{cases} |x_2 - x_1|, & if \quad x_2 \neq x_1, \\ |y_2 - y_1|, & if \quad x_2 = x_1. \end{cases}$$

The geometric location of points equidistant from a given point is called a circle.

In Galilan space , the circle looks as follows

$x^2 = r^2;\ x = \pm r$.

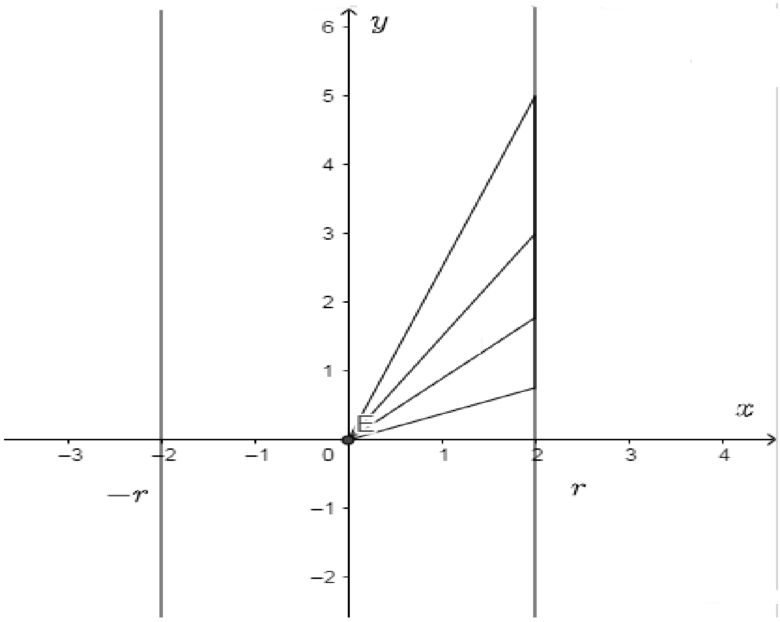

Figure 1.

Given the vectors $\vec{X}(x_1,y_1)$ and $\vec{Y}(x_2,y_2)$ in R_2^1, the unit vectors of these vectors have the following form

$$\tilde{X}\left(1,\frac{y_1}{x_1}\right),\ \tilde{Y}\left(1,\frac{y_2}{x_2}\right);$$

The angle between the vectors $\vec{X}$ and $\vec{Y}$ is calculated as follows

$$h = \left|\frac{y_2}{x_2} - \frac{y_1}{x_1}\right|. \qquad (1)$$

We shoud understand the geometric meaning of the angle formes in the Galilean plane as follows. Let us put the tails of thhe vectors in one point, then draw a unit circle of R_2^1 with the tails of the vectors as the center. The length of the segment formed between the vectors is the angle between these two vectors. (Fig.2)

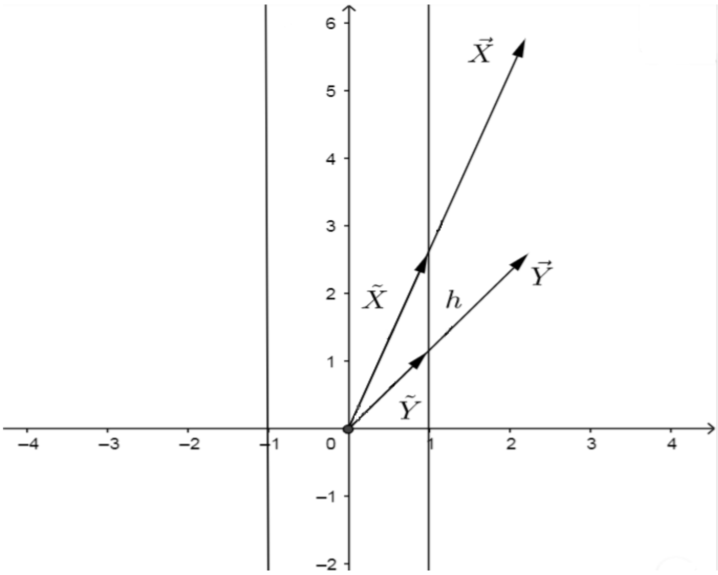

Figure 2.

The Galilean plane is an affine plane in which the distance between two points is defined as the projection of the segment connecting the points on the abscissa axis. If the projection on the abscissa is zero, the distance is equal to the projection of the segment on the ordinate axis.

A line parallel to OY axis in the Galilean space is called a special line. We are interested in finding the area of poly gons whose arbitarary side does not lie in a special line.

Main Result

Problem 1. Given points $A(x_1, y_1)$,$B(x_2, y_2)$ *and* $C(x_3, y_3)$ in a Cartesian coordinate system, find the area of the triangle with vertices A, B, C .

Solution. Let the point $A(x_1, y_1)$,$B(x_2, y_2)$ *and* $C(x_3, y_3)$ are given.

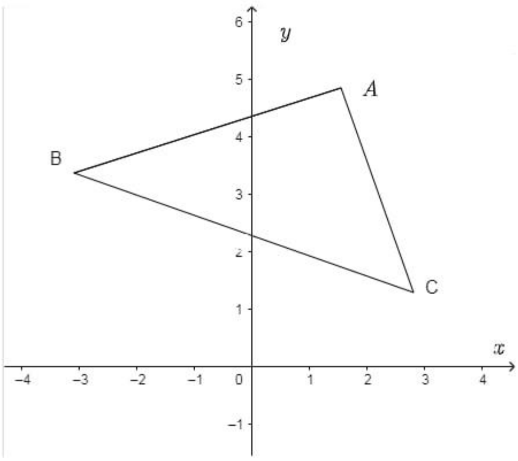

Figure 3.

The area of a triangle is half the product of it is base and height. We fill the triangle ABC to the parallelogram $ABCD$. Diagonals of the parallelogram divides it is area in half, that is, the areas are equilateral triangles.

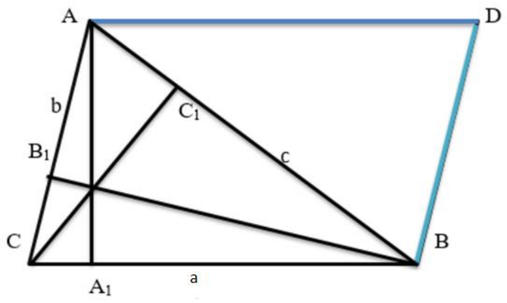

Figure 4.

We know that

$$S_{ABCD}=CB*AA_1=a*h.$$

Therefore

$$S_{ABC}=\frac{1}{2}CB*AA_1=\frac{1}{2}a*h$$

To find this area through the coordinotes, we can construct vectors and use a cross product. We know that the cross product of two vectors is equal to the area of the parallelogram constructed on the vectors, so its half area is equal to the triangle.

Now we construct the vectors $\overrightarrow{AB}=(x_2-x_1,y_2-y_1)$, $\overrightarrow{AC}=(x_3-x_1,y_3-y_1)$ and cross product them to find the area of the triangle ABC

$$S=\frac{1}{2}\left[\overrightarrow{AB},\overrightarrow{AC}\right]=\frac{1}{2}\begin{vmatrix} y_2-y_1 & y_3-y_1 \\ x_2-x_1 & x_3-x_1 \end{vmatrix}$$

$$S=\frac{1}{2}\left[\overrightarrow{AB},\overrightarrow{AC}\right]=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

We can also find the area of a triangle in 3 dimensional Euclidean space. Let the points $A(x_1,y_1,z_1)$ $B(x_2,y_2,z_2)$ $C(x_3,y_3,z_3)$ be given and let a triangle be formed from these points. We can use across product to find it is area:

We construct the vectors

$$\overrightarrow{AB}=(x_2-x_1,y_2-y_1,z_2-z_1) \text{ and } \overrightarrow{AC}=(x_3-x_1,y_3-y_1,z_3-z_1)$$

Thus, the area we need is

$$S=\frac{1}{2}\left[\overrightarrow{AB},\overrightarrow{AC}\right]=\frac{1}{2}\sqrt{(|AB||AC|)^2-(AB\,AC)^2}=\frac{1}{2}\sqrt{A_1+A_2+A_3+A_4+A_5+A_6}$$

Where A_1,A_2,A_3,A_4,A_5,A_6 are found as follows:

$$A_1=(x_2-x_1)(y_2-y_1)\begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_1-x_3 & y_3-y_1 \end{vmatrix} \quad A_2=(x_2-x_1)(z_3-z_1)\begin{vmatrix} x_2-x_1 & z_2-z_1 \\ x_1-x_3 & z_3-z_1 \end{vmatrix}$$

$$A_3 = (x_3 - x_1)(y_2 - y_1)\begin{vmatrix} x_3 - x_1 & z_2 - z_1 \\ x_1 - x_2 & z_3 - z_1 \end{vmatrix} \quad A_4 = (y_2 - y_1)(x_3 - x_1)\begin{vmatrix} y_2 - y_1 & z_2 - z_1 \\ y_1 - y_3 & z_3 - z_1 \end{vmatrix}$$

$$A_5 = (z_2 - z_1)(x_3 - x_1)\begin{vmatrix} z_2 - z_1 & x_2 - x_1 \\ z_1 - z_3 & x_3 - x_1 \end{vmatrix} \quad A_6 = (y_3 - y_1)(z_2 - z_1)\begin{vmatrix} z_2 - z_1 & y_2 - y_1 \\ z_1 - z_3 & y_3 - y_1 \end{vmatrix}.$$

Problem 2.

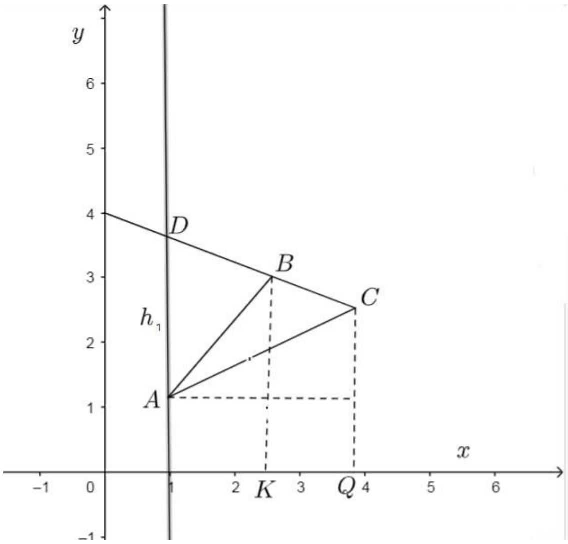

Figure 5.

Given a point A in the Cartesian coordinate system, let us draw a straight line passing A and parallel to Oy axis. Let us take points B and C different from the point A and let the straight line connecting B and C cross the line parallel to Oy and passing A. We denote the distance between A and D with h_1. Let the distance between the points K and Q be $|KQ| = pr_{Ox}AC - pr_{Ox}AB$.

Prove that the following formula holds for the area of the triangle ABC

$$S_{ABC} = \frac{h_1 * |KQ|}{2}$$

Solution. To prove that this formula is true, let us show the several properties for the area of a triangle.

Lemma 1. Given two parallel lines, the points $B_1, B_2, B_3, B_4, \ldots B_n$ lie on the line 1 and the points A and C lie on line 2. We Denote the distance between A and C with $|AC|$. Let the minimal distance between the parallel lines be H,

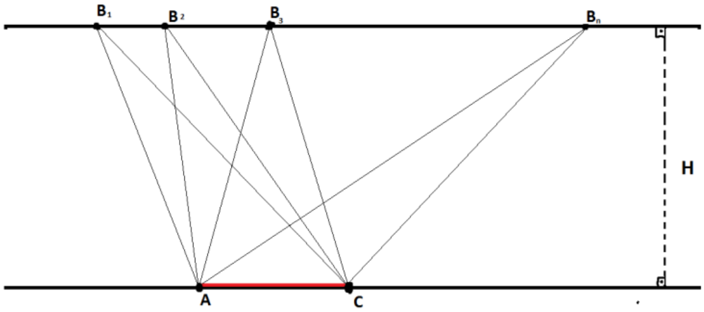

Figure 6

Then we have the formula $S=\frac{|AC|*H}{2}$ for the are of triangle AB_iC. Since H

Is constant triangles $S_{B_1AC}=S_{B_2AC}=S_{B_3AC}=...=S_{B_n\,AC}$ have the same area.

Now, let us rewrite the figure 7 to the following from:

We rotate the coordinate system 90^0 counter clockwise, that is, $Ox\rightarrow Oy$,

$Oy\rightarrow -Ox$. Our next step is to move the points and lines in figure 7 symmetrically across the line $y=x$, so we get:

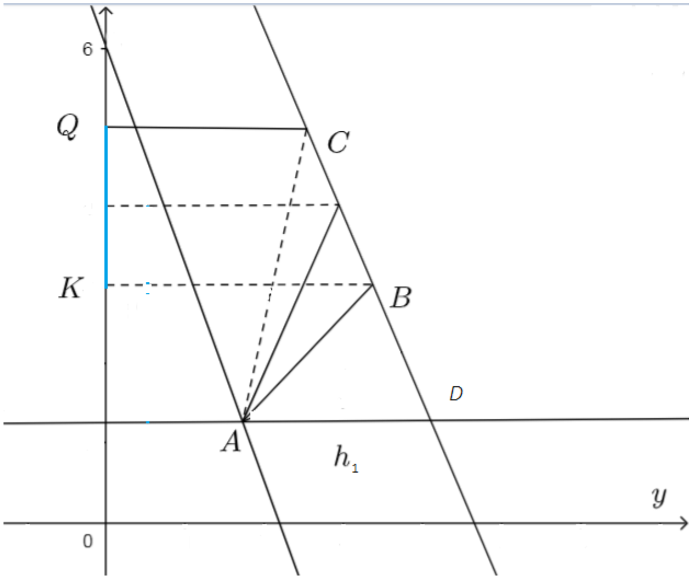

Figure 7.

In addition to this:

1) We draw a line passing through A and parallel to a line passing through the point C and B.

Let $|AZ|$ be the height of the triangle ABC. Then, we move point B on to point D and move point C along that line without changing the distance from

point B. That is, $C\rightarrow C`$, $B\rightarrow D$ and $|CB|=|C`D|$ (Fig 8).

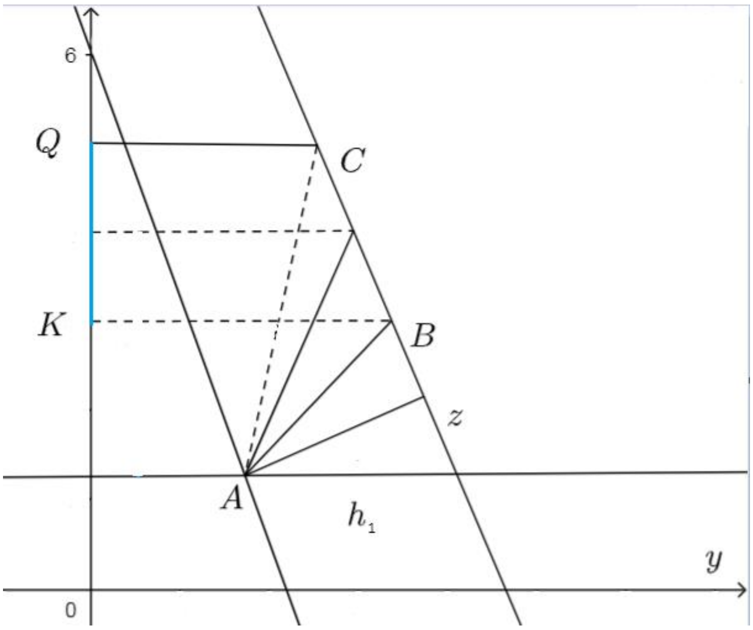

Figure 8.

As a result, point C moves to point $C`$ and point B moves to point D.

Lemma 2. Let us given a straight line that intersects Oy and Ox at A and B_n.

Respectively. We consider the point $B_1, B_2, B_3, \ldots B_{n-1}, B_n$ on that line. Let A_i on of B_i on the Oy axis.

As in the figure 9, under the condition that the angle α does not change, it follows that the distances between adjacent points on the Oy axis do not change. Therefore $|KQ|$ does not change in the figure 8 by Lemma 2.

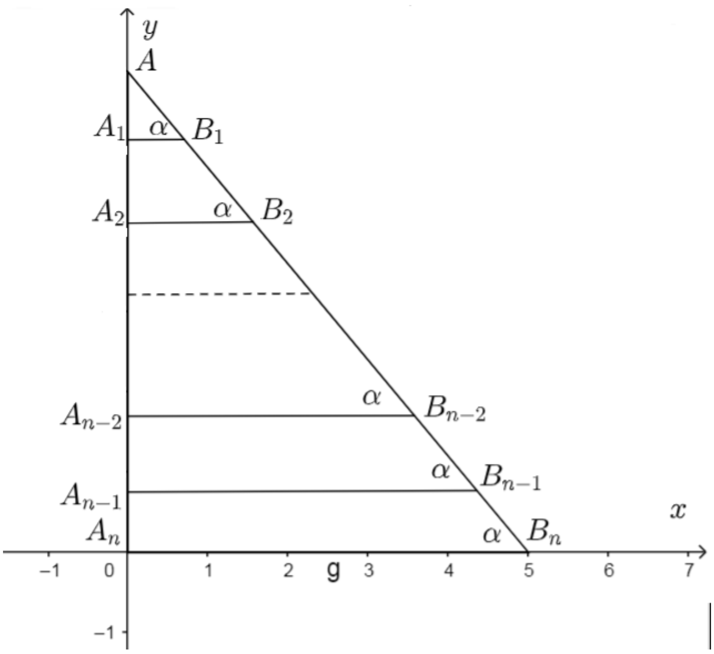

Figure 9.

Using all the above properties and results, we will give the proof of the formula in the final step.

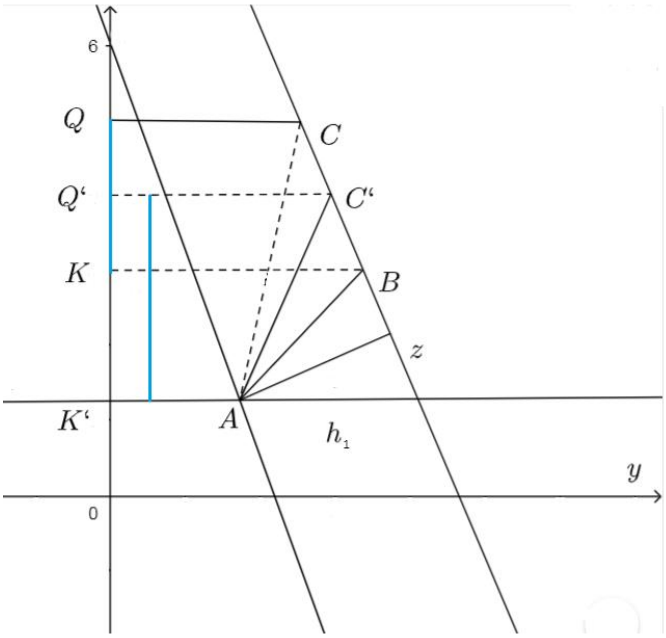

Figure 10.

Results: If we move points C and B along the straight line passing through BC to the points $C`$ and D respectively preserving the distance $|BC|$, the area of the triangle ABC and projection of BC onto the Oy axis will remain un changed.

From there, we find the area using the figure 10.

At the same time, since the area of the triangle does not change on the figure, we move point A and other corresponding points relative to A.

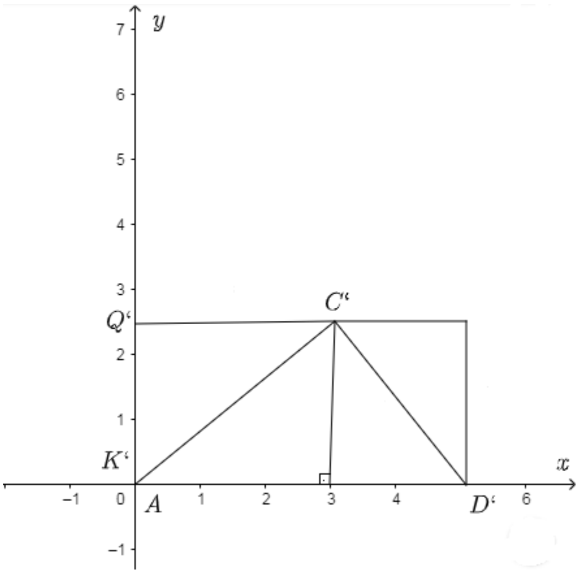

Figure 11.

Using the properties of constancy of the area of the triangle and the length of the projection , formula (5) was obtained, and now, replacing the equal distances we write the formula we need to prove.

Since

$|Q`K`| = |KQ|$, and $|AD| = h_1$, we get

$$S=\frac{|\mathrm{AD}|*|Q`K`|}{2}=\frac{|AD|*|KQ|}{2}=\frac{|KQ|}{2}*h_1$$

Let us give a triangle ABC in Oxy Cartesian coordinate system. Let QV is the segment formed by the intersection of the lines AB and AC along the Oy axis (Fig12.).

Problem 3.

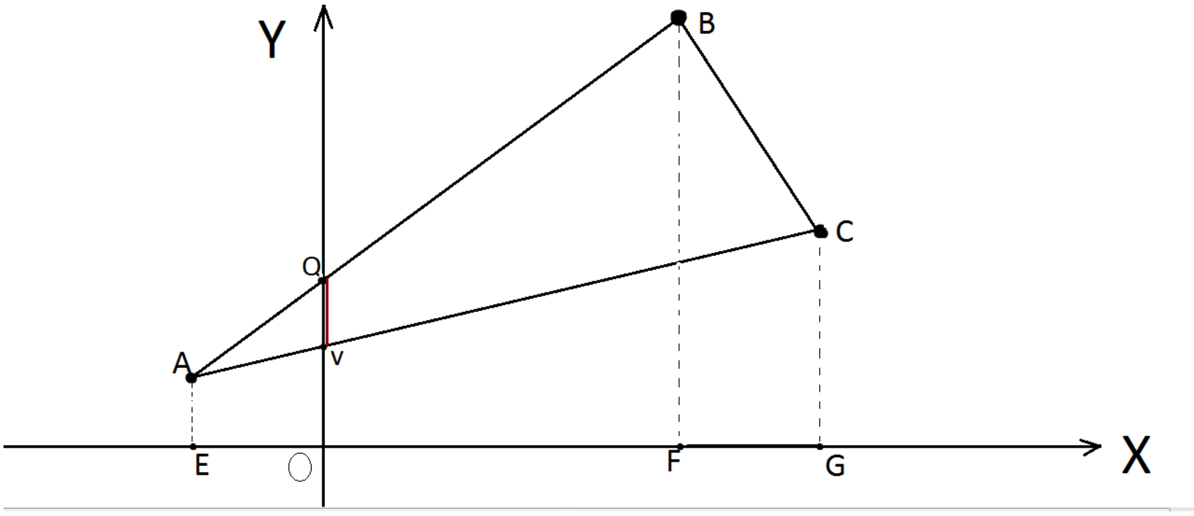

Figure 12.

Let EO, OF and FG be the prosjections of the segment AQ, QB and BC on the Ox axis respectively. Obviously, we have $|QV|=h_1$, $|EG|=pr_{ox}AC=|EO|+|OF|+|FG|$ prove that , if $|EO|=1$ then

$$S_{ABC}=\frac{1}{2}\left(pr_{ox}AB\right)\left(pr_{ox}AC\right)h_1$$

Solution. We prove this formula for arbitrary OE . Now consider the figure 13:

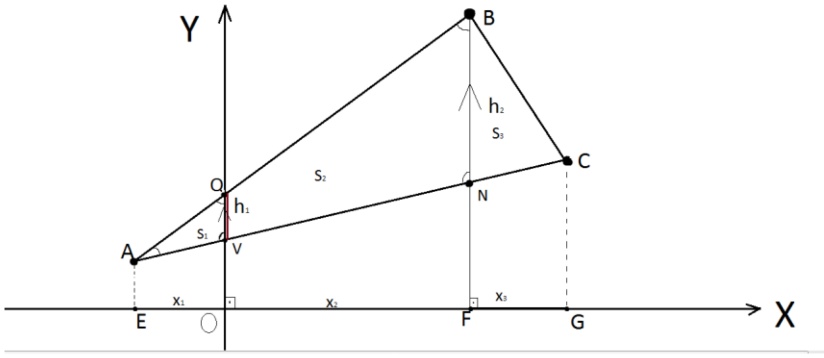

Figure 13.

In this figure 13 we divided triangle ABC in to 2 triangles and a rectangle. Now, let

$OE=X_1$ $OF=X_2$ $S_{ABC}=S_3$ $FG=X_3$ $S_{QBNV}=S_2$ $S_{AQV}=S_1$ $QV=h_1$ $BN=h_2$

Since QV and BN are parallel , it implies that triangles AQV and ABN are similar. We have

$S_1 = \frac{1}{2} x_1 h_1$; $S_3 = \frac{1}{2} x_3 h_3$; and $S_2 = \frac{1}{2}(h_1 + h_2)x_2$

thus

$$S_{ABC} = S_1 + S_2 + S_3 = \frac{1}{2} x_1 h_1 + \frac{1}{2}(h_1 + h_2)x_2 + \frac{1}{2} x_3 h_2 =$$
$$= \frac{1}{2}(x_1 h_1 + (h_1 + h_2)x_2 + x_3 h_2) = \qquad (6)$$
$$= \frac{1}{2}((x_1 + x_2)h_1 + (x_2 + x_3)\ h_2).$$

Using the similarity of the triangles ABN and AQV we can find h_2.

$$\frac{S_1 + S_2}{S_1} = \left(\frac{h_2}{h_1}\right)^2 \Rightarrow S_1 + S_2 = S_1\left(\frac{h_2}{h_1}\right)^2 \Rightarrow S_1\left(\left(\frac{h_2}{h_1}\right)^2 - 1\right) = S_2$$

Since $S_1 = \frac{1}{2} x_1 h_1$; and $S_2 = \frac{1}{2}(h_1 + h_2)x_2$ we get

$$\frac{1}{2} x_1 h_1\left(\left(\frac{h_2}{h_1}\right)^2 - 1\right) = \frac{1}{2}(h_1 + h_2)x_2 \quad \Rightarrow$$
$$x_1 h_1\left(\left(\frac{h_2}{h_1}\right)^2 - 1\right) = (h_1 + h_2)x_2$$
$$h_2^2 - h_1^2 = (h_2 - h_1)(h_2 + h_1)$$

$$\Rightarrow h_2 = h_1\left(\frac{x_1 + x_2}{x_1}\right)$$

There fore, by the formula (6) we get

$$S_{ABC} = \frac{1}{2}(x_1 + x_2)h_1 + (x_2 + x_3)\ h_2 =$$
$$= \frac{1}{2}(x_1 + x_2)h_1 + (x_2 + x_3)h_1\left(\frac{x_1 + x_2}{x_1}\right) =$$
$$= \frac{1}{2}(x_1 + x_2)h_1\left(1 + (x_2 + x_3)\frac{1}{x_1}\right) =$$
$$= \frac{1}{2}(x_1 + x_2)h_1\frac{(x_1 + x_2 + x_3)}{x_1};$$

Then we get

$$S_{ABC} = \frac{1}{2}(x_1 + x_2)h_1\frac{(x_1 + x_2 + x_3)}{x_1} = \frac{1}{2}(pr_{ox}AB)\ (pr_{ox}AC)h_1\frac{1}{x_1}$$

When $|OE| = x_1 = 1$ we get the formula asked in the problem.

$$S_{ABC} = \frac{1}{2}\left(pr_{ox}AB\right)\ \left(pr_{ox}AC\right)h_1\frac{1}{x_1} = \frac{1}{2}\left(pr_{ox}AB\right)\ \left(pr_{ox}AC\right)h_1$$

Conclusion.

During the study of the above theorems and properties, students will get an idea about finding the area of a triangle and about non-Euclidean geometry. In conclusion, let us talk about non- Euclidean geometry. All geometries that differ from Euclidean geometry are called non- Euclidean geometries. For example: Lobachevsky geometry, Riemann geometry, Galilean geometry, etc. In total there are 9 geometries in the plane and 27 geometries in 3-dimensional space.

Reference.

1. Artykbaev A. Sokolov D.D. Geometry as a whole in flat space-time. - Tashkent: “Fan”, 1990.
2. Jaglom I.M. The principle of relativity of of Galilean and non-Euclidean geometry. Nauka, Moscow, (1996).
3. Khachaturyan A.V. Geometry of Galileo Moscow 2005.
4. Artykbaev A., Sultanov B.M. Invariants of Surface Indicatrix in a Special Linear Transformation. *Mathematics and Statistics* 7(4). 2019. pp. 106-115. DOI: 10.13189/ms.2019.070403
5. Makarova N. M. “On the theory of cycles of parabolic geometry on the plane”, Sibirsk. math. zhurn., 2:1 (1961), 68–81 23
6. Bakelman I.Ya., Werner A.L., Kontor B.E. Introduction to differential geometry “in general” // Moscow, Science. (1973). - 440 pp.
7. Blaschke V. Introduction to differential geometry. – U.: Udmurt University Publishing House, Regular and Chaotic Dynamics, 2005. - 232 pp.