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Differential Equations Third Order Inhomogeneous Linear with Boundary Conditions

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ABSTRACT

Considering the importance of teaching linear differential equations, it can be said that every physical and technical phenomenon, which is expressed and modeled in mathematical sciences, is a differential equation. Differential equations are essential part of contemporary comparative mathematics that covers all fields of physics (heat, mechanics, atoms, electronics, magnetism, light and waves), many economic subjects, engineering subjects, natural problems, population growth and technical problems today. In this article, we will consider the theory of linear inhomogeneous differential equations of the third order with boundary conditions and the transformation of coefficients into multiple $p(x)$ functions. In the field of differential equations, a boundary value problem with a set of additional constraints is called boundary value problem. The solution of this boundary value problem is actually a solution for the differential equation with the given constraints, which actually satisfies the conditions of the boundary value problem. Differential equation problems with boundary conditions are similar to initial value problems. A boundary value problem with conditions defined on the boundaries is an independent variable in the equation, while an initial value problem is defined as the same condition that has the value of the independent variable and this value is less than the limit, hence the term value is initial and the initial value is the amount of data that matches the minimum or maximum input, internal, or output value specified for a system or component. When the boundaries of the boundary values in the solution of obtaining the constants of the third order differential equation D_1, D_2 and D_3 are determined, the failure to obtain the constants is called the boundary problem. We solve this problem by considering the given conditions for the real Green's function. Every real function is a solution of a set of linear differential equations, and the values of its boundary value depend on the intervals.

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1. INTRODUCTION

Differential equations are one of the most interesting and widely used mathematical topics that have attracted the attention of many researchers. Differential equations are used in various fields including physics; It is especially useful in the movement of weights attached to it, springs, electric circuits, and free vibrations. In mathematics, in the field of differential equations, the boundary value problem of a differential equation with an additional set of constraints is called the boundary condition problem, and the solution that satisfies the given conditions also satisfies [1,2,3]. boundary value problems arise in several branches of physics because each equation has a differential body. Wave equation problems, such as determining normal modes, are often referred to as boundary value problems. Another big group of important boundary value problems are Sturm-Liouville problems. The analysis of these problems includes special functions and Green's functions of a differential equation [3, 4].

In this article, discussions are discussed to express and understand more about the problem of third-order inhomogeneous linear differential equations with boundary conditions and obtaining Green's function.

In the space $L_2(0,1)$ of an inhomogeneous linear differential equation of the third order with boundary conditions, we consider the boundary problem and on that initial function is obtained by a linear differential equation of the third order with constant coefficients [2].

Suppose we consider the third order differential equation as follows:

$$y^{(3)}(x)+P_1(x)y^{(1)}(x)+P_0(x)y(x)=f(x) \qquad (1)$$

Where $P_0(x)$ and $P_1(x)$ many functions on intervals $[0,1]$, the number 3 expresses the order of the differential equation.

In this section, we consider the known features of these functions with the following boundary conditions:

$$U_j(y)=\alpha_j y^{(\gamma_j)}(0)+\beta_j y^{(\gamma_j)}(1)=0, \quad j=1,2,3 \quad (2)$$

It should be known that

$$\gamma_1=0,\ \gamma_2=1,\ \gamma_3=2$$

is remembered.

From the statement of the above problem, the following questions can be reached:

Research questions

1. How can we solve the inhomogeneous linear differential equation of the third order $y^{(3)}(x)+P_1(x)y^{(1)}(x)=f(x)$ with boundary conditions $U_j(y)=y^{(\gamma_j)}(0)+y^{(\gamma_j)}(1)=0, j=1,2,3$?

2. How can we solve the private and general solution of the inhomogeneous third-order linear differential equation $y^{(3)}(x)+P_2(x)y^{(2)}(x)+P_1(x)y^{(1)}(x)+P_0(x)y(x)=f(x)$ if it is with $P_2(x)\neq 0,\ P_1(x)\neq 0,\ P_0(x)\neq 0$ and given boundary conditions $U_j(y)=y^{(\gamma_j)}(0)+y^{(\gamma_j)}(1)=0, \quad j=1,2,3$?

3. How can we solve the inhomogeneous linear differential equation of the third order with the conditions of the boundary problem $U_j(y)=y^{(\gamma_j)}(0)+y^{(\gamma_j)}(1)=0, \quad j=1,2,3$ in the determined area?

The opinion framework is based on third order differential equations with boundary conditions, Green's function, eigenfunctions and eigenvalues to obtain general solution and private solution on boundary conditions, Wronski determinant and differential operators.

This research is divided into six basic parts: introduction, review of scientific works, basic concepts, research findings, controversy and conclusion.

2. Literature review

Differential equations have been developed for nearly 300 years, and the relationship between the transformations of functions and the derivatives of functions, so its history naturally goes back to the discovery of the derivative by the English scientist Isaac Newton between the years (1642-1772). And the German Gottfried Leibniz worked on differential equations, including first-order differential equations, in the years (1646-1716). Jacob proposed Bernoulli's differential equation in 1674, but he was unable to prove it until Euler proved it in 1705. Sturm-Liouville theorized the boundary problem with the first boundary in linear differential equations. And its applications, the classical Sturm-Liouville theory, named after Jacques Francois Sturm and Joseph Liouville, was proposed between (1855-1803) and in (1809-1882), the theory of linear differential equations was formed in the second order. In 1969, the Russian scientist Naimark wrote in his book Linear Differential Functions about the Green's function for solving differential equations with boundary conditions. According to the theorems of Mikhailov and Kesselman, the boundary conditions are often strictly regular and defined [4]. Therefore, the eigenvalues of the asymptotic operator are simple and distinct, there is a positive number such as δ, which are separated from each other by a greater distance δ for both eigenvalues of the function [3]. It is also concluded from the works [1, 2, 3, 7, 11, 12, 13, 14, 15] that the system of eigenfunctions and related functions form a basis Res in the space.

In recent years, many pure mathematical scientists have worked in the field of obtaining Green's function for linear differential equations, including the Kazakh scientist Kanguzhin in 2019, who published an article entitled "Getting Green's function for second-order linear differential equations" [4, 6].

3. Elementary Basic

The general form of inhomogeneous linear differential equations can be written as follows, considering differential operators:

$$L(y)=\lambda y(x)+f(x) \qquad (3)$$

Considering the system of high-order linear differential equations of the general solution of equation (1), (2), we can consider an initial function as follows:

$$y(x)=y_0(x)+D_1\varphi_1(x)+D_2\varphi_2(x)+D_3\varphi_3(x) \quad (4)$$

Where

$$y_0(x)=\int_0^x g(x,t)f(t)dt$$

$y_0(x)$ is the homogeneous solution of the above equation and $\varphi_1(x),\ \varphi_2(x),\ \varphi_3(x)$ The main system of solving the equation with homogeneous conditions $L(\varphi_1)=0,\ L(\varphi_2)=0,\ L(\varphi_3)=0$ is one of the inhomogeneous boundary conditions $\varphi_j^{(k-1)}(0)=\delta_{kj}$ function $g(x,t)$ It is determined by the following formula, which can be called Green's function [9].

$$g(x,t)=\frac{P(x,t)}{W(t)}$$

Where $\delta_{kj}=\begin{cases}1\ , k=j\\ 0\ , k\neq j\end{cases}$ and $W(t)$ determinant Wronski

$$W(t_1,t_2,t_3)=\begin{vmatrix} y_1(t) & y_2(t) & y_3(t)\\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t)\\ y_1^{(2)}(t) & y_2^{(2)}(t) & y_3^{(2)}(t)\end{vmatrix}$$

And it should be known that $P(x,t)$ is equal to:

$$P(x,t)=\begin{vmatrix} y_1(t) & y_2(t) & y_3(t)\\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t)\\ y_1(x) & y_2(x) & y_3(x)\end{vmatrix}.$$

So you should know that $g(x,t)=P(x,t)$ so $g(x,t)$ can be defined from the following formula.

$$g(x,t)=\begin{vmatrix} y_1(t) & y_2(t) & y_3(t)\\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t)\\ y_1(x) & y_2(x) & y_3(x)\end{vmatrix}.$$

From here we can propose a specific inhomogeneous solution as follows

$$y_0(x)=\int_0^x \begin{vmatrix} y_1(t) & y_2(t) & y_3(t)\\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t)\\ y_1(x) & y_2(x) & y_3(x)\end{vmatrix} f(t)dt$$

The inhomogeneous solution function $y_0(x)$ is equation (1), (2) and for its correctness, I search the first, second and third order derivatives and establish the proposed third order equation (1).

$$y_0^{(1)}(x)=\int_0^x \begin{vmatrix} y_1(t) & y_2(t) & y_3(t)\\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t)\\ y_1^{(1)}(x) & y_2^{(1)}(x) & y_3^{(1)}(x)\end{vmatrix} f(t)dt+\begin{vmatrix} y_1(x) & y_2(x) & y_3(x)\\ y_1^{(1)}(x) & y_2^{(1)}(x) & y_3^{(1)}(x)\\ y_1(x) & y_2(x) & y_3(x)\end{vmatrix} f(x)$$

Now we take the second derivative

$$y_0^{(2)}(x)=\int_0^x \begin{vmatrix} y_1(t) & y_2(t) & y_3(t)\\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t)\\ y_1^{(2)}(x) & y_2^{(2)}(x) & y_3^{(2)}(x)\end{vmatrix} f(t)dt$$

Now, in the same way, we get the derivative of the third order

$$y_0^{(3)}(x)=\int_0^x \begin{vmatrix} y_1(t) & y_2(t) & y_3(t) \\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t) \\ y_1^{(3)}(x) & y_2^{(3)}(x) & y_3^{(3)}(x) \end{vmatrix} f(t)dt + \begin{vmatrix} y_1(x) & y_2(x) & y_3(x) \\ y_1^{(1)}(x) & y_2^{(1)}(x) & y_3^{(1)}(x) \\ y_1^{(2)}(x) & y_2^{(2)}(x) & y_3^{(2)}(x) \end{vmatrix} f(x)$$

Now, for the correctness of the received function, we must establish and check the price of the function and its derivatives of different degrees in equation (1).

$$L(y)=y_0{}^{(3)}(x)+P_1(x)y_0{}^{(1)}(x)+P_0(x)y_0(x)$$

$$L(y)=\int_0^x \begin{vmatrix} y_1(t) & y_2(t) & y_3(t) \\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t) \\ y_1^{(3)}(x) & y_2^{(3)}(x) & y_3^{(3)}(x) \end{vmatrix} f(t)dt + f(x) +$$

$$+P_1(x)\int_0^x \begin{vmatrix} y_1(t) & y_2(t) & y_3(t) \\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t) \\ y_1^{(1)}(x) & y_2^{(1)}(x) & y_3^{(1)}(x) \end{vmatrix} f(t)dt$$

$$+P_0(x)\int_0^x \begin{vmatrix} y_1(t) & y_2(t) & y_3(t) \\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t) \\ y_1(x) & y_2(x) & y_3(x) \end{vmatrix} f(t)dt$$

From here we add the determinants together,

$$L(y)=$$

$$=\int_0^x \begin{vmatrix} y_1(t) & y_2(t) & y_3(t) \\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t) \\ y_1^{(3)}(x)+P_1(x)y_1^{(1)}(x)+P_0(x)y_1(x) & y_2^{(3)}(x)+P_1(x)y_2^{(1)}(x)+P_0(x)y_2(x) & y_3^{(3)}(x)+P_1(x)y_3^{(1)}(x)+P_0(x)y_3(x) \end{vmatrix}$$

$$f(t)dt+f(x).$$

Conditions of homogeneous equation $L(y)=y_1^{(3)}(x)+P_1(x)y_1^{(1)}(x)+P_0(x)y_1(x)=0$ so that we can solve function of $f(x)$ and as a result we can say that we have obtained the solution of the inhomogeneous part.

We have obtained the Green's function for the proposed problem and according to the problem, we have proved that:

$$L(y)=f(x), \qquad 0<x<1 \tag{5}$$

with boundary conditions

$$U_1(y)=0\,,\ U_2(y)=0,\ U_3(y)=0\,. \tag{6}$$

The type of boundary conditions that are already defined for us.

$$U_1(y)=\alpha_1 y(0)-\beta_1 y(1)=0$$

$$U_2(y)=\alpha_2 y'(0)-\beta_2 y'(1)=0$$

$$U_3(y)=\alpha_3 y''(0)-\beta_3 y''(1)=0$$

It can be said that we can solve the equation and Green's function (5), (6) using Green's functions as follows.

$$y(x,t)=(L_0-\lambda I)^{-1} f=\int_0^1 G_0(x,t,\lambda)f(t)dt$$

where

$$G_0(x,t,\lambda) = -\frac{\begin{vmatrix} y_1(x,\lambda) & y_2(x,\lambda) & y_3(x,\lambda) & g(x,t) \\ U_1(y_1) & U_1(y_2) & U_1(y_3) & U_1(g) \\ U_2(y_1) & U_2(y_2) & U_2(y_3) & U_2(g) \\ U_3(y_1) & U_3(y_2) & U_3(y_3) & U_3(g) \end{vmatrix}}{\begin{vmatrix} U_1(\mathrm{y}_1) & U_1(\mathrm{y}_2) & U_1(\mathrm{y}_3) \\ U_2(\mathrm{y}_1) & U_2(\mathrm{y}_2) & U_2(\mathrm{y}_3) \\ U_3(\mathrm{y}_1) & U_3(\mathrm{y}_2) & U_3(\mathrm{y}_3) \end{vmatrix}}$$

$G_0(x,t,\lambda) -$ is a Green's function.

we assume

$3 \geq \gamma_2 \geq \gamma_1 \geq \gamma_0 \geq 0$

4. Main results

$$y(x,t) = (L_0 - \lambda I)^{-1} f = -\int_0^1 \frac{\begin{vmatrix} y_1(x,\lambda) & y_2(x,\lambda) & y_3(x,\lambda) & g(x,t) \\ U_1(y_1) & U_1(y_2) & U_1(y_3) & U_1(g) \\ U_2(y_1) & U_2(y_2) & U_2(y_3) & U_2(g) \\ U_3(y_1) & U_3(y_2) & U_3(y_3) & U_3(g) \end{vmatrix}}{\begin{vmatrix} U_1(\mathrm{y}_1) & U_1(\mathrm{y}_2) & U_1(\mathrm{y}_3) \\ U_2(\mathrm{y}_1) & U_2(\mathrm{y}_2) & U_2(\mathrm{y}_3) \\ U_3(\mathrm{y}_1) & U_3(\mathrm{y}_2) & U_3(\mathrm{y}_3) \end{vmatrix}} f(t)dt$$

If $x > t$ the function $g(x,t)$ has the following form

$$g(x,t) = \begin{vmatrix} y_1(t) & y_2(t) & y_3(t) \\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t) \\ y_1(x) & y_1(x) & y_1(x) \end{vmatrix}$$

If $x \leq t$ is then a function $g(x,t) = 0$.

$$\Delta_0(\lambda) = \begin{vmatrix} U_1(\mathrm{y}_1) & U_1(\mathrm{y}_2) & U_1(\mathrm{y}_3) \\ U_2(\mathrm{y}_1) & U_2(\mathrm{y}_2) & U_2(\mathrm{y}_3) \\ U_3(\mathrm{y}_1) & U_3(\mathrm{y}_2) & U_3(\mathrm{y}_3) \end{vmatrix}$$

5. Discussion

From the topic of research, we come to the conclusion that the problem we studied in inhomogeneous linear differential equations of the third order is a set of Green's function. Every real function exists in the solution of a set of linear differential equations, and such equations have not only one definite solution but also several solutions. Its field of application in physics, for example, finding the temperature at all points of an iron rod with one end at absolute zero and the other end at the freezing point of water, is a boundary value problem.

If the problem depends on both space and time, the value of the problem can be determined at a certain point for all times or at a certain time for the entire space and provide another example of a linear differential equation with boundary conditions.

The boundary condition that determines the value of the function is the Dirichlet boundary condition. For example, if one end of an iron rod is held at absolute zero, the magnitude of the problem is determined at that point in space.

6. Conclusion

Since we have obtained the Green's function for solving the third-order inhomogeneous linear differential equation, everything in this system is technically solvable. To solve it, we proposed the received method and showed that the inhomogeneous linear differential equation of the third order with the boundary conditions of the problem does not have one solution, but has several solutions in terms of eigenvalues and eigenfunctions.

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